Journal of Advanced Analysis and Applications
E-ISSN: 3125-9251
https://jaaa-journal.org
Volume 1, Issue 1 (2026), Pages 69–92
DOI: https://doi.org/10.66745/jaaa.v1i1.005

# Functional Contractions in Relational Metric Spaces

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**Abstract.** The 2015 fixed point result on rs-relational metric spaces, due to Alam and Imdad [J. Fixed Point Theory Appl. 17 (2015), 693–702], is equivalent to the classical Banach contraction principle [Fund. Math. 3 (1922), 133–181]. The same is true for the 1961 statement in metric spaces, due to Edelstein [Proc. Amer. Math. Soc. 12 (1961), 7–10], and for the 2005 fixed point result in quasi-ordered metric spaces obtained by Nieto and Rodríguez-López [Order 22 (2005), 223–239].

**Keywords:** relational metric space; monotone mapping; fixed point; global strongly Picard operator; Banach contraction principle
**MSC (2020):** 47H10, 54H25

## 1 Introduction

Let $X$ be a nonempty set. A subset $Y$ of $X$ is called an almost singleton (in short, an asingleton) if

$$y_1, y_2 \in Y \implies y_1 = y_2.$$

It is called a singleton if, in addition, $Y$ is nonempty. In this case,

$$Y = \{y\},$$

for some $y \in X$.

Take a metric

$$d : X \times X \to \mathbb{R}_+ = [0, \infty[$$

on $X$, as well as a self-map $T \in \mathcal{F}(X)$. Here, for each pair $(A, B)$ of nonempty sets, $\mathcal{F}(A, B)$ denotes the class of all functions from $A$ to $B$. When $A = B$, we write $\mathcal{F}(A)$ instead of $\mathcal{F}(A, A)$. Denote

$$\text{Fix}(T) = \{x \in X : x = Tx\}.$$

Each point of this set is called a fixed point of $T$.

The basic existence and uniqueness criteria involving such points will be discussed together with some precise concepts. These concepts are essentially suggested by the developments in Rus [23, Ch. 2, Sect. 2.2] and consist of three distinct stages, described below.

**(St-1)** The uniqueness context is based on the following convention:

(uni) We say that $T$ is fix-asingleton if $\text{Fix}(T)$ is an asingleton, and fix-singleton if $\text{Fix}(T)$ is a singleton.

**(St-2)** The convergence context consists of two metric concepts:

(pic-1) We say that $T$ is a Picard operator modulo $d$ if, for each $x \in X$, the sequence

$$T^N x = (T^n x)_{n \geq 0}$$

is $d$-Cauchy. We say that $T$ is a global Picard operator modulo $d$ if, in addition, $T$ is fix-asingleton.

(pic-2) We say that $T$ is a strongly Picard operator modulo $d$ if, for each $x \in X$, the sequence $T^N x$ is $d$-convergent and

$$\lim_n T^n x \in \mathrm{Fix}(T).$$

We say that $T$ is a global strongly Picard operator modulo $d$ if, in addition, $T$ is fix-asingleton or, equivalently, fix-singleton.

**(St-3)** The contractive setting consists of certain predicate-type constructions involving the data $(X, d; T)$ and has the general form

(contr)
$$(\forall x, y \in X) : \quad \Pi_1(d; x, y) \Longrightarrow \Pi_2(d; T^N x, T^N y).$$

Here, $\Pi_1$ and $\Pi_2$ are propositional constructions.

By combining these three stages, we obtain the vast majority of results belonging to metrical fixed point theory. The simplest result in this area may be stated as follows.

We first define the following concepts:

(Con-1) We say that $T$ is $(d; \alpha)$-contractive, where $\alpha \in \mathbb{R}_+$, if

$$d(Tx, Ty) \leq \alpha d(x, y), \qquad \text{for all } x, y \in X.$$

(Con-2) We say that $T$ is $(d; 0, 1)$-contractive if it is $(d; \alpha)$-contractive for some $\alpha \in [0, 1[$.

**Theorem 1.1.** *Let $(X, d)$ be a metric space, and let $T$ be a self-map of $X$. Assume that $(X, T)$ is $d$-Banach, that is, $X$ is $d$-complete and $T$ is $(d; 0, 1)$-contractive. Then $T$ is a global strongly Picard operator modulo $d$.*

Technically speaking, the proof of this result is essentially contained in the classical 1922 statement due to Banach [3]. Thus, it is natural to call it the Banach contraction principle on metric spaces (in short, (B-cp-ms)). This result has found a multitude of applications in operator equation theory; therefore, it has been the subject of many extensions.

Among these extensions, we mention the implicit relational approach to generalizing (B-cp-ms), based on set-implicit contractive conditions of the form

(si-gen)
$$\big(d(Tx, Ty), d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(Tx, y)\big) \in \mathcal{M},$$

for all $x, y \in X$ such that $x\mathcal{R}y$.

Here, $\mathcal{M}$ is a subset of $\mathbb{R}_+^6$, and $\mathcal{R} \subseteq X \times X$ is a relation on $X$.

In particular, when

$$\mathcal{R} = X \times X,$$

a basic variant of the general contractive property above is

(si-pla)
$$\big(d(Tx,Ty),d(x,y)\big) \in \mathcal{M},$$
for all $x, y \in X$.

Here, $\mathcal{M} \subseteq \mathbb{R}_+^2$ is a subset of the plane.

Classical examples in this direction are due to Meir and Keeler [16], Cirić [6], and Matkowski [15]. Other classical contributions in this area are due to Boyd and Wong [4], Leader [12], Matkowski [14], and Reich [20]. For different extensions of these results, we refer to the survey papers by Rhoades [21], Park [18], and Collaco and E Silva [7].

Furthermore, when

(q-ord) $\mathcal{R}$ is a quasi-order that is, a reflexive and transitive relation on $X$,

a first result in this area was obtained by Turinici in 1986 [25]. Two decades later, this fixed point statement was rediscovered by Ran and Reurings [19] and Agarwal et al. [1].

Finally, under the assumption

(amorph) $\mathcal{R}$ is amorphous, that is, it has no specific properties on $X$,

a first pair of results was obtained by Jachymski in 2008 [9], in the setting of metric spaces endowed with a graph, and by Samet and Turinici in 2012 [24], in the setting of relational metric spaces. In fact, as shown by Roldán and Shahzad [22], these two approaches are identical. Further extensions of the latter result were obtained by Alam and Imdad in 2015 [2].

Having made these points precise, our first aim is to establish, in a genuine relational context, a functional extension of the Alam–Imdad result by reducing it to a simpler asymptotic statement derived from the 2003 developments of Kirk [11]. Then, passing to the linear case, we show that the rs-relational version of the quoted Alam–Imdad result is equivalent to the Banach contraction principle on metric spaces.

In addition, we stress that this equivalence also includes the 1961 fixed point result in metric spaces due to Edelstein [8], as well as the 2005 related statement in quasi-ordered metric spaces due to Nieto and Rodríguez-López [17]. Further aspects will be treated elsewhere.

## 2 Preliminaries

Let $X$ be a nonempty set. Let $d(\cdot,\cdot)$ be a metric on $X$, and let $\mathcal{R} \subseteq X \times X$ be a reflexive relation on $X$. The triple $(X, d, \mathcal{R})$ is called a relational metric space. We further denote

(rs-cov)
$$\mathcal{S} = \mathcal{R} \cup \mathcal{R}^{-1}$$
the reflexive symmetric cover of $\mathcal{R}$;

(rt-cov)
$$\mathcal{R}^\omega = \bigcup\{\mathcal{R}^n : n \geq 1\}$$
the reflexive transitive cover of $\mathcal{R}$.

An equivalent definition of the latter relation is as follows. Given $x, y \in X$ and $k \geq 2$, any element
$$A = (z_1, \dots, z_k) \in X^k$$

satisfying

$$z_1 = x, \qquad z_k = y, \qquad z_i \mathcal{R} z_{i+1}, \quad i \in \{1, \ldots, k-1\},$$

is called a $k$-dimensional $(\mathcal{R})$-chain between $x$ and $y$. Its associated $d$-length is

$$d(A) = d(z_1, z_2) + \cdots + d(z_{k-1}, z_k).$$

The class of all such chains will be denoted by $X_k(x, y; \mathcal{R})$. Further, put

$$X(x, y; \mathcal{R}) = \bigcup \{X_k(x, y; \mathcal{R}) : \ k \geq 2\}.$$

Any element of this set is called an $(\mathcal{R})$-chain in $X$ joining $x$ and $y$.

The announced relation is now described as follows:

$$(x, y \in X) : \quad x(\mathcal{R}^{\omega})y \quad \text{if and only if} \quad X(x, y; \mathcal{R}) \text{ is nonempty}.$$

Further, let $\mathcal{B}$ be a relation on $X$. A subset $Y$ of $X$ is called a $(\mathcal{B})$-almost-singleton (in short, a $(\mathcal{B})$-asingleton) if

$$y_1, y_2 \in Y, \quad y_1 \mathcal{B} y_2 \quad \Longrightarrow \quad y_1 = y_2.$$

It is called a $(\mathcal{B})$-singleton if, in addition, $Y$ is nonempty.

Given a sequence $(z_n)$ in $X$ and a point $z \in X$, we define the ascending property

(asc) $(z_n)$ is $(\mathcal{B})$-ascending if

$$z_i \mathcal{B} z_{i+1},$$

for each index $i$.

We also define the following boundedness-type concepts:

(bd-1) $(z_n)\mathcal{B}z$ if

$$z_n \mathcal{B} z, \qquad \text{for all } n;$$

(bd-2) $(z_n)\mathcal{B}\mathcal{B}z$ if

$$(w_n)\mathcal{B}z,$$

for some subsequence $(w_n)$ of $(z_n)$.

Finally, let $T$ be a self-map of $X$. We have to determine conditions under which $\text{Fix}(T)$ is nonempty. To this end, we start from the basic hypothesis

(incr) $T$ is $(\mathcal{R})$-increasing:

$$x\mathcal{R}y \quad \Longrightarrow \quad Tx\mathcal{R}Ty.$$

Note that the natural condition to be added here is

(s-prog) $T$ is $(\mathcal{R})$-semi-progressive:

$$X(T, \mathcal{R}) = \{x \in X : \ x\mathcal{R}Tx\} \neq \emptyset.$$

However, even if this condition fails, the problem remains meaningful in a vacuous sense.

The basic directions under which our investigations will be conducted are described in the list below, which is comparable with the one in Turinici [26]:

**rpic-0)** We say that $T$ is fix-($\mathcal{B}$)-asingleton if $\text{Fix}(T)$ is ($\mathcal{B}$)-asingleton, and fix-($\mathcal{B}$)-singleton if $\text{Fix}(T)$ is ($\mathcal{B}$)-singleton.

**rpic-1)** We say that $T$ is a Picard operator modulo $(d, \mathcal{R})$ if, for each $x \in X(T, \mathcal{R})$,

$$(T^n x)_{n \geq 0}$$

is $d$-Cauchy. We say that $T$ is a ($\mathcal{B}$)-global Picard operator modulo $(d, \mathcal{R})$ if, in addition, $T$ is fix-($\mathcal{B}$)-asingleton.

**rpic-2)** We say that $T$ is a strongly Picard operator modulo $(d, \mathcal{R})$ if, for each $x \in X(T, \mathcal{R})$,

$$(T^n x)_{n \geq 0}$$

is $d$-convergent and

$$\lim_n T^n x \in \text{Fix}(T).$$

We say that $T$ is a ($\mathcal{B}$)-global strongly Picard operator modulo $(d, \mathcal{R})$ if, in addition, $T$ is fix-($\mathcal{B}$)-asingleton or, equivalently, fix-($\mathcal{B}$)-singleton.

The regularity conditions for such properties are based on the ascending property introduced above. For the moment, we list two concepts.

**reg-1)** We say that $X$ is $(d, \mathcal{R})$-complete if

$$(z_n) \text{ is } (\mathcal{R})\text{-ascending and } d\text{-Cauchy}$$

implies that

$$(z_n) \text{ is } d\text{-convergent in } X.$$

**reg-2)** We say that $\mathcal{R}$ is $d$-almost-self-closed if

$$(z_n) \text{ is } (\mathcal{R})\text{-ascending and } z_n \xrightarrow{d} z$$

implies that

$$(z_n)\mathcal{R}\mathcal{R}z.$$

**Remark 2.1.** Concerning the last condition, there are two stronger variants with practical meaning:

(reg-2a) We say that $\mathcal{R}$ is $d$-self-closed if

$$(z_n) \text{ is } (\mathcal{R})\text{-ascending and } z_n \xrightarrow{d} z$$

implies that

$$(z_n)\mathcal{R}z.$$

(reg-2b) We say that $\mathcal{R}$ is $d$-almost-closed if

$$z_n \xrightarrow{d} z$$

implies that

$$(z_n)\mathcal{R}\mathcal{R}z.$$

Clearly,

(incl-1)

$$\mathcal{R} \text{ is } d\text{-self-closed}$$

implies that

$$\mathcal{R} \text{ is } d\text{-almost-self-closed;}$$

(incl-2)

$$\mathcal{R} \text{ is } d\text{-almost-closed}$$

implies that

$$\mathcal{R} \text{ is } d\text{-almost-self-closed.}$$

However, the relationship between the assumptions on the left-hand sides of these implications remains, for the moment, an open problem.

As an essential completion of these developments, we introduce the contractive conditions to be used. Denote by $\mathcal{F}_0(\mathbb{R}_+)$ the class of all functions $\varphi \in \mathcal{F}(\mathbb{R}_+)$ such that $\varphi(0) = 0$. Then, let $\mathcal{F}_0(in, re)(\mathbb{R}_+)$ denote the class of all functions $\varphi \in \mathcal{F}_0(\mathbb{R}_+)$ that are increasing and regressive, that is,

$$\varphi(t) < t, \qquad \forall t \in \mathbb{R}_+^0.$$

The following properties of these functions will be used:

(M-ad) $\varphi$ is Matkowski admissible [14] if each sequence $(t_n)_{n \geq 0}$ in $\mathbb{R}_+^0$ satisfying

$$t_{n+1} \leq \varphi(t_n), \qquad \forall n,$$

fulfills

$$\lim_n t_n = 0.$$

The class of all such functions will be denoted by

$$\mathcal{F}_0(in, re; Mat)(\mathbb{R}_+).$$

(B-ad) $\varphi$ is Browder admissible [5] if each sequence $(t_n)_{n \geq 0}$ in $\mathbb{R}_+^0$ satisfying

$$t_{n+1} \leq \varphi(t_n), \qquad \forall n,$$

fulfills

$$\sum_n t_n < \infty.$$

The class of all such functions will be denoted by

$$\mathcal{F}_0(in, re; Bro)(\mathbb{R}_+).$$

It follows directly from the definitions that

$$\mathcal{F}_0(in, re; Bro)(\mathbb{R}_+) \subseteq \mathcal{F}_0(in, re; Mat)(\mathbb{R}_+).$$

However, the converse inclusion is not true in general.

Now, for an arbitrary fixed $\varphi \in \mathcal{F}_0(\mathbb{R}_+)$, we define the following concept:

(R-phi) $T$ is $(d,\mathcal{R};\varphi)$-contractive if

$$d(Tx,Ty) \le \varphi(d(x,y)), \qquad \forall x,y \in X, \quad x\mathcal{R}y.$$

Note that, by the properties of the metric $d(\cdot,\cdot)$, this condition yields the following one:

(S-phi) $T$ is $(d,\mathcal{S};\varphi)$-contractive if

$$d(Tx,Ty) \le \varphi(d(x,y)), \qquad \forall x,y \in X, \quad x\mathcal{S}y.$$

In particular, when

$\varphi$ is linear, that is,

$$\varphi(t) = \lambda t, \qquad t \in \mathbb{R}_+,$$

for some $\lambda \ge 0$,

it will be useful to introduce the following simplifying conventions:

(lin-1) $T$ is $(d,\mathcal{R};\varphi)$-contractive will be written as

$$T \text{ is } (d,\mathcal{R};\lambda)\text{-contractive;}$$

(lin-2) $T$ is $(d,\mathcal{S};\varphi)$-contractive will be written as

$$T \text{ is } (d,\mathcal{S};\lambda)\text{-contractive.}$$

Note that, under the choice $\lambda = 1$, we have

(nex-1) $T$ is $(d,\mathcal{R};1)$-contractive means that

$$d(Tx,Ty) \le d(x,y), \qquad x,y \in X, \quad x\mathcal{R}y.$$

In this case, we say that

$$T \text{ is } (d,\mathcal{R})\text{-nonexpansive.}$$

(nex-2) $T$ is $(d,\mathcal{S};1)$-contractive means that

$$d(Tx,Ty) \le d(x,y), \qquad x,y \in X, \quad x\mathcal{S}y.$$

In this case, we say that

$$T \text{ is } (d,\mathcal{S})\text{-nonexpansive.}$$

Finally, we need the following meta-convention. Given a generic contraction principle (CP-gen), denote by

$$((\text{CP-gen}))$$

the universe of (CP-gen), that is, the class of all contraction principles deducible from (CP-gen).

In this case, given two contraction principles (CP-1) and (CP-2),

(incl)

$$((\text{CP-1})) \subseteq ((\text{CP-2}))$$

means that (CP-1) is deducible from (CP-2);

(eq)

$$((\text{CP-1})) = ((\text{CP-2}))$$

means that (CP-1) is equivalent to (CP-2).

A basic example of this type, suggested by Kirk's 2003 paper [11], is given below. Some additional regularity conditions are needed.

**(ARC-1)** Let $[d;T]$ and $[[d;T]]$ denote the relations on $X$ given by

(a-rela)

$$x[d;T]y \quad \text{if and only if} \quad \lim_n d(T^n x, T^n y) = 0$$

the asymptotic relation associated with $(d,T)$;

(ta-rela)

$$x[[d;T]]y \quad \text{if and only if} \quad \sum_n d(T^n x, T^n y) < \infty$$

the telescopic asymptotic relation associated with $(d,T)$.

Clearly, $[d;T]$ and $[[d;T]]$ are equivalence relations on $X$, with

$$[[d;T]] \subseteq [d;T].$$

Having made these points precise, we define the following concept:

(s-asy) $T$ is strongly $(d,\mathcal{B})$-asymptotic if

$$\mathcal{B} \subseteq [[d;T]].$$

**Remark 2.2.** Two basic properties involving this concept are useful.

**(I)** The following singleton-type property is valid:

$$T \text{ is strongly } (d,\mathcal{B})\text{-asymptotic}$$

implies that

$$T \text{ is fix-}(\mathcal{B})\text{-asingleton.}$$

Indeed, let $z_1, z_2 \in \text{Fix}(T)$ with $(z_1, z_2) \in \mathcal{B}$. Then

$$d(z_1, z_2) = \lim_n d(T^n z_1, T^n z_2) = 0.$$

Hence, $z_1 = z_2$.

**(II)** In addition, the following relative property holds:

$$T \text{ is strongly } (d,\mathcal{R})\text{-asymptotic}$$

if and only if

$$T \text{ is strongly } (d,\mathcal{S}^\omega)\text{-asymptotic.}$$

The verification is immediate, in view of

$$\mathcal{R} \subseteq [[d;T]] \Longrightarrow \mathcal{S} \subseteq [[d;T]] \Longrightarrow \mathcal{S}^{\omega} \subseteq [[d;T]] \Longrightarrow \mathcal{R} \subseteq [[d;T]].$$

**(ARC-2)** Given the structure $(X, d, \mathcal{R})$ and the self-map $T$ of $X$, we define the following concept:

$T$ is left $(d, \mathcal{R})$-continuous if, for each sequence $(u_n)$ in $X$ and each $u \in X$,

$$u_n \xrightarrow{d} u \quad \text{and} \quad (u_n)\mathcal{R}u$$

imply that

$$Tu_n \xrightarrow{d} Tu.$$

**Remark 2.3.** A concrete situation in which this property holds is given by the following implication:

$$T \text{ is } (d, \mathcal{R})\text{-nonexpansive}$$

implies that

$$T \text{ is left } (d, \mathcal{R})\text{-continuous.}$$

Indeed, let $(u_n)$ be a sequence in $X$ and let $u \in X$ be as in the premise above. From the nonexpansive condition, we have

$$d(Tu_n, Tu) \le d(u_n, u), \qquad \text{for all } n.$$

This, together with the convergence property, gives the desired conclusion.

We may now formulate a basic statement in this area, referred to as the Kirk asymptotic fixed point theorem in relational metric spaces (in short, (K-asy-rms)).

**Theorem 2.4.** *Let the structure $(X, d, \mathcal{R})$ and the self-map $T$ of $X$ be such that $(X, T)$ is $(d, \mathcal{R})$-Kirk, in the sense that:*

(i) *$T$ is strongly $(d, \mathcal{R})$-asymptotic and left $(d, \mathcal{R})$-continuous;*

(ii) *$X$ is $(d, \mathcal{R})$-complete and $\mathcal{R}$ is $d$-almost-self-closed;*

(iii) *$T$ is $(\mathcal{R})$-increasing and $(\mathcal{R})$-semi-progressive.*

*Then, $T$ is an $(\mathcal{S}^{\omega})$-global strongly Picard operator modulo $(d, \mathcal{R})$.*

*Proof.* By a preceding remark,

$T$ is strongly $(d, \mathcal{S}^{\omega})$-asymptotic; hence, $T$ is fix-$(\mathcal{S}^{\omega})$-asingleton.

It remains to prove that $T$ is strongly Picard modulo $(d, \mathcal{R})$.

Take some $x_0 \in X(T, \mathcal{R})$ and put

$$x_n = T^n x_0, \qquad n \ge 0.$$

Clearly, $(x_n)$ is $(\mathcal{R})$-ascending. From the strongly $(d, \mathcal{R})$-asymptotic property with $x = x_0$ and $y = x_1$, we have

$$\sum_n d(x_n, x_{n+1}) < \infty.$$

Hence, $(x_n)$ is an $(\mathcal{R})$-ascending $d$-Cauchy sequence. Since $X$ is $(d, \mathcal{R})$-complete,

$$x_n \xrightarrow{d} z,$$

for some $z \in X$. Further, since $\mathcal{R}$ is $d$-almost-self-closed, there exists a subsequence

$$u_n = x_{i(n)}, \qquad n \geq 0,$$

of $(x_n)_{n\geq 0}$, hence satisfying

$$u_n \xrightarrow{d} z,$$

such that

$$(u_n)\mathcal{R}z.$$

This, together with the fact that $T$ is left $(d, \mathcal{R})$-continuous, gives

$$Tu_n \xrightarrow{d} Tz.$$

On the other hand,

$$Tu_n = x_{i(n)+1}, \qquad n \geq 0,$$

is a subsequence of $(x_n)_{n\geq 0}$. Hence,

$$Tu_n \xrightarrow{d} z.$$

Combining this with the uniqueness of the $d$-limit, we obtain

$$d(z, Tz) = 0.$$

Therefore, $z = Tz$, as desired.

□

**Remark 2.5.** By a simple verification, it follows that

$$((\text{B-cp-ms})) \subseteq ((\text{K-asy-rms})).$$

Indeed, it is enough to take $\mathcal{R} = X \times X$ in this relational statement.

Whether the reciprocal inclusion

$$((\text{K-asy-rms})) \subseteq ((\text{B-cp-ms}))$$

also holds is an open problem; we conjecture that the answer is negative.

## 3 Functional reflexive case

Let $(X, d)$ be a metric space. Further, let $\mathcal{R}$ be a reflexive relation on $X$; then, $(X, d, \mathcal{R})$ will be referred to as a relational metric space.

The following statement, referred to as the Alam–Imdad functional contraction principle in relational metric spaces (in short, (AI-fct-rms)), is our starting point.

**Theorem 3.1.** *Let the structure $(X, d, \mathcal{R})$ and the self-map $T$ of $X$ be such that*

(i) *$T$ is $(d, \mathcal{R}; \varphi)$-contractive for some*

$$\varphi \in \mathcal{F}_0(in, re; Bro)(\mathbb{R}_+);$$

(ii) *$X$ is $(d, \mathcal{R})$-complete and $\mathcal{R}$ is $d$-almost-self-closed;*

(iii) *$T$ is $(\mathcal{R})$-increasing and $(\mathcal{R})$-semi-progressive.*

*Then, $T$ is an $(\mathcal{S}^\omega)$-global strongly Picard operator modulo $(d, \mathcal{R})$.*

This result may be viewed as a functional version of the 2015 statement of Alam and Imdad [2]. In fact, the Alam–Imdad result is a relational variant of the 2008 result of Jachymski [9], established in metric spaces endowed with a graph. This assertion follows from the developments of Roldán and Shahzad [22], according to which the graph setting can be directly converted into a relational one.

Concerning the status of this result, the following is valid.

**Proposition 3.2.** *Under these conventions, we have*

$$((\text{AI-fct-rms})) \subseteq ((\text{K-asy-rms})).$$

*This means that*

*(AI-fct-rms) is deducible from (K-asy-rms).*

*Proof.* There are two steps.

**Step 1.** First, we prove that

$$T \text{ is strongly } (d, \mathcal{R})\text{-asymptotic;}$$

hence, by the preceding discussion,

$$T \text{ is strongly } (d, \mathcal{S}^\omega)\text{-asymptotic.}$$

In fact, let $(x, y) \in \mathcal{R}$ be arbitrary. Since $T$ is $(d, \mathcal{R}; \varphi)$-contractive, we have

$$d(T^n x, T^n y) \leq \varphi^n(d(x, y)), \qquad \text{for all } n.$$

Hence,

$$\sum_n d(T^n x, T^n y) \leq \sum_n \varphi^n(d(x, y)) < \infty,$$

because $\varphi$ is Browder admissible. Thus, $T$ is strongly $(d, \mathcal{R})$-asymptotic.

**Step 2.** By the choice of $\varphi$ and a previous fact, we have

$$T \text{ is } (d, \mathcal{R}; \varphi)\text{-contractive} \Longrightarrow T \text{ is } (d, \mathcal{R})\text{-nonexpansive}$$

and

$$T \text{ is } (d,\mathcal{R})\text{-nonexpansive} \Longrightarrow T \text{ is left } (d,\mathcal{R})\text{-continuous.}$$

Putting these facts together, we obtain that $(X,T)$ is $(d,\mathcal{R})$-Kirk. Therefore, the conclusion of (AI-fct-rms) follows directly from (K-asy-rms). □

A natural problem is whether the functional result above remains valid when the function

$$\varphi \in \mathcal{F}_0(in,re)(\mathbb{R}_+)$$

is only Matkowski admissible. In the genuine relational context, this is not possible in general. However, when $\mathcal{R}$ is a quasi-order, that is, a reflexive and transitive relation, some positive answers to this problem are available. A basic statement of this type may be obtained along the lines of Agarwal et al. [1]. Further implicit-type aspects may be obtained by following the developments of Turinici [27] and the references therein.

## 4 Linear rs-relational case

Let $(X,d)$ be a metric space. Further, let $\mathcal{R}$ be a reflexive relation on $X$; then, $(X,d,\mathcal{R})$ will be referred to as a relational metric space.

As shown above, the functional result (AI-fct-rms) is reducible to the simpler principle (K-asy-rms). Since this latter principle is not known to be deducible from (B-cp-ms), a similar conclusion may be derived for our starting functional result. This remains valid even when one passes to its linear version:

(lin) (AI-lin-rms) = the result (AI-fct-rms) with $\varphi$ linear.

It is then natural to ask whether, in the particular case where

$\mathcal{R}$ is reflexive and symmetric hence identical with $\mathcal{S}$,

this underlying conclusion is still retained. Fortunately, the answer is negative, in the following sense: the functional statement above can be reduced to (B-cp-ms) under such a linear symmetric context.

Let $(X,d)$ be a metric space. Further, let $\mathcal{S}$ be a reflexive symmetric relation on $X$; then, $(X,d,\mathcal{S})$ will be referred to as an rs-relational metric space.

The following linear symmetric version of the functional statement above, referred to as the Alam–Imdad linear contraction principle on rs-relational metric spaces (in short, (AI-lin-rsms)), naturally enters our discussion.

**Theorem 4.1.** *Let the structure $(X,d,\mathcal{S})$ and the self-map $T$ of $X$ be such that*

(i) *$T$ is $(d,\mathcal{S};\lambda)$-contractive for some $\lambda \in ]0,1[$;*

(ii) *$X$ is $(d,\mathcal{S})$-complete and $\mathcal{S}$ is $d$-almost-self-closed;*

(iii) *$T$ is $(\mathcal{S})$-increasing and $(\mathcal{S}^\omega)$-semi-progressive.*

*Then, $T$ is an $(\mathcal{S}^\omega)$-global strongly Picard operator modulo $(d,\mathcal{S}^\omega)$.*

Concerning the status of this principle, the following is valid.

**Proposition 4.2.** *Under the precise context, we have*

$$((\text{B-cp-ms})) \subseteq ((\text{AI-lin-rsms})) \subseteq ((\text{B-cp-ms})).$$

*Hence, the statements (B-cp-ms) and (AI-lin-rsms) are mutually equivalent.*

*Proof.* The first inclusion

$$((\text{B-cp-ms})) \subseteq ((\text{AI-lin-rsms}))$$

is clear: it is enough to take $\mathcal{S} = X \times X$ in (AI-lin-rsms).

It remains to prove that the second inclusion

$$((\text{AI-lin-rsms})) \subseteq ((\text{B-cp-ms}))$$

also holds. Assume that the conditions of (AI-lin-rsms) are satisfied.

For simplicity, denote

$$\mathcal{E} = \mathcal{S}^{\omega}.$$

Clearly, $\mathcal{E}$ is an equivalence relation on $X$.

By the developments in the preceding section, it is clear that

> $T$ is strongly $(d, \mathcal{E})$-asymptotic; hence, by the discussion above, $T$ is fix-$(\mathcal{E})$-asingleton.

It remains to prove that

> $T$ is a strongly Picard operator modulo $(d, \mathcal{E})$: for each $x_0 \in X$ with $x_0(\mathcal{E})Tx_0$, there exists $z \in \text{Fix}(T)$ such that the iterative sequence
>
> $$x_n = T^n x_0, \qquad n \geq 0,$$
>
> satisfies
>
> $$x_n \xrightarrow{d} z \quad \text{as } n \to \infty.$$

There are several steps.

**Step 1.** Let $x_0 \in X$ be as above, and denote

$$X_0 = X(x_0, \mathcal{E}) = \{y \in X : x_0(\mathcal{E})y\};$$

that is, the equivalence class of $x_0$ with respect to $\mathcal{E}$.

Some elementary properties of this class are listed in the following lemma.

**Lemma 4.3.** *Under the precise context, we have:*

(p-1) *For each $x \in X_0$,*

$$X(x, \mathcal{S}) \subseteq X(x, \mathcal{E}) \subseteq X_0.$$

(p-2) *$X_0$ is $(\mathcal{E})$-connected: for each $x, y \in X_0$, we have $x(\mathcal{E})y$.*

(p-3) *Any $(\mathcal{S})$-chain in $X$ between two points of $X_0$ is an $(\mathcal{S})$-chain in $X_0$ between the same points;*

(p-4) *$X_0$ is $(d, \mathcal{S})$-closed: the $d$-limit of any $(\mathcal{S})$-ascending, $d$-convergent sequence in $X_0$ belongs to $X_0$.*

*Proof.* (p-1), (p-2): These properties are evident.

(p-3): Given $x, y \in X_0$, let

$$A = (z_1, \ldots, z_k) \in X^k$$

be a $k$-dimensional $(\mathcal{S})$-chain in $X$ between $x$ and $y$. The case $k = 2$ is clear; hence, without loss of generality, we may assume that $k \geq 3$. Clearly,

$$(\forall i \in \{2, \ldots, k-1\}) : \quad x(\mathcal{E})z_i.$$

Since $x_0(\mathcal{E})x$, it follows that $z_i \in X_0$ for all $i \in \{2, \ldots, k-1\}$.

(p-4): Let $(z_n)$ be a sequence in $X_0$ and let $z \in X$ be such that $(z_n)$ is $(\mathcal{S})$-ascending and

$$z_n \xrightarrow{d} z \quad \text{as } n \to \infty.$$

Since $\mathcal{S}$ is $d$-almost-self-closed, there exists a subsequence

$$w_n = z_{i(n)}, \qquad n \geq 0,$$

of $(z_n)_{n\geq 0}$ such that

$$(w_n)\mathcal{S}z.$$

Hence, for each fixed $n \geq 0$, we have

$$x_0\mathcal{E}w_n \quad \text{and} \quad w_n\mathcal{E}z.$$

Therefore, $x_0\mathcal{E}z$, which means that $z \in X_0$. The conclusion follows. □

**Step 2.** We introduce a pseudometric

$$e : X_0 \times X_0 \to \mathbb{R}_+$$

as follows: for each $x, y \in X_0$,

$$e(x, y) = \inf \{d(A) : A \in X(x, y; \mathcal{S})\},$$

where, for

$$A = (z_1, \ldots, z_k) \in X^k, \qquad k \geq 2,$$

we have

$$d(A) = d(z_1, z_2) + \cdots + d(z_{k-1}, z_k),$$

and $A$ is an $(\mathcal{S})$-chain in $X$, hence in $X_0$, between $x$ and $y$. The definition is consistent since $X_0$ is $(\mathcal{E})$-connected.

Some basic properties of this object are contained in the following lemma.

**Lemma 4.4.** *Under the introduced setting, we have the following assertions:*

(Asr-1) $e(\cdot, \cdot)$ *is a metric on* $X_0$ *that subordinates d, meaning that*

$$d(x, y) \leq e(x, y), \qquad (x, y) \in X_0 \times X_0.$$

(Asr-2) *Moreover, the* $(\mathcal{S})$*-identity relation holds:*

$$e(x, y) = d(x, y),$$

*whenever* $x, y \in X_0$ *and* $x(\mathcal{S})y$.

*Proof.* By its definition, $e$ is reflexive,

$$e(x,x) = 0, \qquad \forall x \in X_0,$$

symmetric,

$$e(x,y) = e(y,x), \qquad \forall x,y \in X_0,$$

and triangular,

$$e(x,z) \leq e(x,y) + e(y,z), \qquad \forall x,y,z \in X_0.$$

In addition, the triangular property of $d$ gives

$$d(x,y) \leq d(A) = d(z_1,z_2) + \cdots + d(z_{k-1},z_k),$$

for every $k \geq 2$ and every $(\mathcal{S})$-chain $A = (z_1,\ldots,z_k)$ in $X$, hence in $X_0$, between $x$ and $y$. Passing to the infimum, we obtain

$$d(x,y) \leq e(x,y), \qquad \forall x,y \in X_0.$$

Thus, $d$ is subordinated to $e$.

As a direct consequence, $e$ is sufficient:

$$e(x,y) = 0 \Longrightarrow x = y.$$

Hence, $e$ is a metric on $X_0$.

Finally, if $x,y \in X_0$ and $x(\mathcal{S})y$, then $(x,y)$ is an $(\mathcal{S})$-chain between $x$ and $y$. Therefore, by the definition of $e$,

$$e(x,y) \leq d(x,y).$$

Combining this with the already proved inequality

$$d(x,y) \leq e(x,y),$$

we obtain

$$e(x,y) = d(x,y).$$

The proof is complete. □

**Step 3.** We next discuss the completeness of the metric space $(X_0,e)$.

**Lemma 4.5.** *Under the same context, we have*

*$X_0$ is $e$-complete, that is, every $e$-Cauchy sequence in $X_0$ converges in $X_0$.*

*Proof.* Let $(u_n)$ be an $e$-Cauchy sequence in $X_0$. By definition, there exists a strictly increasing sequence of indices $(k(n))_{n\geq 0}$ such that

$$(\forall n): \quad k(n) \leq m \quad \Longrightarrow \quad e(u_{k(n)},u_m) < 2^{-n}.$$

Denoting

$$v_n = u_{k(n)}, \qquad n \geq 0,$$

we therefore have

$$e(v_n,v_{n+1}) < 2^{-n}, \qquad \forall n.$$

Moreover, by the imposed $e$-Cauchy property, $(u_n)$ is $e$-convergent if and only if $(v_n)$ is $e$-convergent. To establish this last property, we proceed as follows.

Since $e(v_0, v_1) < 2^0$, there exists, with the initial index $p(0) = 0$, an $(\mathcal{S})$-chain

$$(z_{p(0)}, \dots, z_{p(1)})$$

in $X$, hence in $X_0$, between $v_0$ and $v_1$ so that

$$p(1) - p(0) \geq 1, \qquad z_{p(0)} = v_0, \qquad z_{p(1)} = v_1,$$

and

$$d(z_{p(0)}, z_{p(0)+1}) + \cdots + d(z_{p(1)-1}, z_{p(1)}) < 2^0.$$

Further, since $e(v_1, v_2) < 2^{-1}$, there exists an $(\mathcal{S})$-chain

$$(z_{p(1)}, \dots, z_{p(2)})$$

in $X$, hence in $X_0$, between $v_1$ and $v_2$ so that

$$p(2) - p(1) \geq 1, \qquad z_{p(1)} = v_1, \qquad z_{p(2)} = v_2,$$

and

$$d(z_{p(1)}, z_{p(1)+1}) + \cdots + d(z_{p(2)-1}, z_{p(2)}) < 2^{-1}.$$

Continuing in this way, and using the $(\mathcal{S})$-identity relation, we obtain an $(\mathcal{S})$-ascending sequence $(z_n)_{n\geq 0}$ in $X_0$ such that

$$\sum_n e(z_n, z_{n+1}) = \sum_n d(z_n, z_{n+1}) < \sum_n 2^{-n} < \infty.$$

Hence,

$$(z_n) \text{ is both } e\text{-Cauchy and } d\text{-Cauchy}.$$

By the second property, since $X$ is $(d, \mathcal{S})$-complete and $X_0$ is $(d, \mathcal{S})$-closed, we obtain

$$z_n \xrightarrow{d} z \quad \text{as } n \to \infty,$$

for some $z \in X_0$. Combining this with the fact that $\mathcal{S}$ is $d$-almost-self-closed, there exists a subsequence

$$t_n = z_{q(n)}, \qquad n \geq 0,$$

of $(z_n)_{n\geq 0}$ such that

$$(t_n)\mathcal{S}z.$$

First, by the $d$-convergence above, we have

$$t_n \xrightarrow{d} z \quad \text{as } n \to \infty.$$

Second, by the $(\mathcal{S})$-identity relation,

$$e(t_n, z) = d(t_n, z), \qquad \forall n.$$

Therefore,

$$t_n \xrightarrow{e} z \quad \text{as } n \to \infty.$$

On the other hand, as already noted, $(z_n)$ is $e$-Cauchy. Together with the $e$-convergence of the subsequence $(t_n)$, this gives

$$z_n \xrightarrow{e} z \quad \text{as } n \to \infty.$$

Since

$$z_{p(n)} = v_n, \qquad n \geq 0,$$

we also have

$$v_n \overset{e}{\longrightarrow} z \quad \text{as } n \to \infty.$$

Consequently, $(u_n)$ is $e$-convergent to $z \in X_0$. The proof is complete. □

**Step 4.** Finally, we discuss some properties of the ambient self-map $T$ of $X$.

**Lemma 4.6.** *The following assertions are valid:*

(asr-1) *$T$ is $(\mathcal{E})$-increasing on $X$:*

$$u, v \in X, \quad u(\mathcal{E})v \quad \Longrightarrow \quad Tu(\mathcal{E})Tv.$$

(asr-2) *$X_0$ is $T$-invariant:*

$$T(X_0) \subseteq X_0.$$

(asr-3) *the restriction of $T$ to $X_0$ is $(e;\lambda)$-contractive:*

$$e(Tu, Tv) \leq \lambda e(u, v), \qquad \forall u, v \in X_0.$$

*Proof.* (asr-1): Given $u, v \in X$ with $u(\mathcal{E})v$, let

$$(z_1, \ldots, z_k), \qquad k \geq 2,$$

be an $(\mathcal{S})$-chain in $X$ connecting them. Since $T$ is $(\mathcal{S})$-increasing,

$$(Tz_1, \ldots, Tz_k)$$

is an $(\mathcal{S})$-chain in $X$ between $Tu$ and $Tv$. Therefore, by definition, $Tu(\mathcal{E})Tv$.

(asr-2): In particular, when $u = x_0$, we have

$$x_0(\mathcal{E})v,$$

meaning that $v \in X_0$, implies

$$Tx_0(\mathcal{E})Tv.$$

Together with $x_0(\mathcal{E})Tx_0$, this gives

$$x_0(\mathcal{E})Tv.$$

Hence, $Tv \in X_0$.

(asr-3): Let $u, v \in X_0$ be given. By the preceding step,

$$u(\mathcal{E})v \qquad \text{and} \qquad Tu, Tv \in X_0.$$

Further, let

$$(z_1, \ldots, z_k), \qquad k \geq 2,$$

be an $(\mathcal{S})$-chain in $X$, hence in $X_0$, connecting $u$ and $v$. Since $T$ is an $(\mathcal{S})$-increasing self-map of $X$,

$$(Tz_1, \ldots, Tz_k)$$

is an $(\mathcal{S})$-chain in $X$, hence in $X_0$, between $Tu$ and $Tv$. Using the contractive condition, we obtain

$$e(Tu,Tv) \leq \sum_{i=1}^{k-1} d(Tz_i, Tz_{i+1}) \leq \lambda \sum_{i=1}^{k-1} d(z_i, z_{i+1}).$$

Since this holds for every such $(\mathcal{S})$-chain, passing to the infimum gives

$$e(Tu,Tv) \leq \lambda e(u,v).$$

By the arbitrariness of the pair $(u, v)$, the restriction of $T$ to $X_0$ is $(e; \lambda)$-contractive. □

**Step 5.** Summing up, (B-cp-ms) is applicable to the metric space $(X_0, e)$ and to the restriction of $T$ to $X_0$. Therefore, by its conclusion,

$T$ is fix-asingleton and strongly Picard modulo $e$

on $X_0$. The second part tells us that, for the initial point $x_0 \in X_0$, there exists $z \in \mathrm{Fix}(T) \cap X_0$ such that the iterative sequence

$$x_n = T^n x_0, \qquad n \geq 0,$$

satisfies

$$e(x_n, z) \to 0 \quad \text{as } n \to \infty.$$

Since $d$ is subordinated to $e$, we necessarily have

$$d(x_n, z) \to 0 \quad \text{as } n \to \infty.$$

Putting these facts together, the proof is complete. □

Note that further extensions of these results are obtainable in the class of generalized metric spaces considered by Luxemburg [13] and Jung [10]. We will discuss these facts elsewhere.

## 5 Particular aspects

In what follows, we give two basic particular cases of this result.

**Part-Case-1)** Let $(X, d)$ be a metric space. Given $\varepsilon > 0$, let $[d < \varepsilon]$ denote the relation on $X$ defined by

$$(x, y \in X): \quad x[d < \varepsilon]y \quad \text{if and only if} \quad d(x,y) < \varepsilon.$$

Clearly, $[d < \varepsilon]$ is reflexive and symmetric.

Further properties of this relation are contained in the following proposition.

**Proposition 5.1.** *Under the introduced setting, the following assertions hold:*

(Aser-1) *$X$ is $(d, [d < \varepsilon])$-complete if and only if $X$ is $d$-complete, that is,*

*each $d$-Cauchy sequence is $d$-convergent*;

(Aser-2) *$[d < \varepsilon]$ is $d$-almost-closed; hence, it is $d$-almost-self-closed;*

(Aser-3) *for each self-map $T$ of $X$ and each $\lambda \in ]0,1[$,*

$$T \text{ is } (d, [d < \varepsilon]; \lambda)\text{-contractive}$$

*implies that*

$$T \text{ is } [d < \varepsilon]\text{-increasing}.$$

*Proof.* (Aser-1): The implication from right to left is clear. For the converse implication, let $(x_n)$ be a $d$-Cauchy sequence in $X$. From the imposed condition, for our fixed $\varepsilon > 0$, there exists an index $k = k(\varepsilon)$ such that

$$d(x_n, x_m) < \varepsilon,$$

hence

$$x_n[d < \varepsilon]x_m,$$

whenever $k \leq n \leq m$. The translated subsequence

$$y_n = x_{k+n}, \qquad n \geq 0,$$

of $(x_n)_{n\geq 0}$ is therefore $[d < \varepsilon]$-ascending and $d$-Cauchy. By the starting hypothesis, $(y_n)$ is $d$-convergent. Hence, $(x_n)$ is $d$-convergent, as desired.

(Aser-2): Let $(x_n)$ be a sequence in $X$, and let $z \in X$ be such that

$$x_n \xrightarrow{d} z \quad \text{as } n \to \infty.$$

By the convergence property, for this $\varepsilon > 0$, there exists an index $q = q(\varepsilon)$ such that

$$(\forall n \geq q): \quad d(x_n, z) < \varepsilon.$$

This means that

$$x_n[d < \varepsilon]z, \qquad \forall n \geq q.$$

Thus, the translated subsequence

$$y_n = x_{q+n}, \qquad n \geq 0,$$

of $(x_n)_{n\geq 0}$ satisfies

$$(y_n)[d < \varepsilon]z.$$

The assertion follows.

(Aser-3): Let $x, y \in X$ be such that $x[d < \varepsilon]y$, that is,

$$d(x, y) < \varepsilon.$$

By the contractive property, we derive

$$d(Tx, Ty) \leq \lambda d(x, y) < \lambda\varepsilon < \varepsilon.$$

This means that

$$Tx[d < \varepsilon]Ty.$$

The conclusion follows. □

Putting these facts together, the following statement, referred to as the Edelstein contraction principle on metric spaces (in short, (E-cp-ms)), enters our discussion.

**Theorem 5.2.** *Let the metric space $(X, d)$, the number $\varepsilon > 0$, and the self-map $T$ of $X$ be taken as follows, under the proposed conventions:*

(i) *$T$ is $(d, [d < \varepsilon]; \lambda)$-contractive for some $\lambda \in ]0, 1[$;*

(ii) *$X$ is $d$-complete and $T$ is $[d < \varepsilon]^{\omega}$-semi-progressive.*

*Then, $T$ is a $[d < \varepsilon]^{\omega}$-global strongly Picard operator modulo $(d, [d < \varepsilon]^{\omega})$.*

To clarify the status of this principle, we need the following Banach contraction principle on bounded metric spaces (in short, (B-cp-bdms)).

**Theorem 5.3.** *Let the metric space $(X, d)$ and the self-map $T$ of $X$ be such that*

*$(X, T)$ is bounded $d$-Banach: $X$ is bounded and $d$-complete, and $T$ is $(d; 0, 1)$-contractive.*

*Then, $T$ is a global strongly Picard operator modulo $d$.*

We may now give an appropriate answer to the posed question.

**Proposition 5.4.** *Under the precise context, we have*

$$((\text{B-cp-ms})) \subseteq ((\text{B-cp-bdms})) \subseteq ((\text{E-cp-ms})) \subseteq ((\text{AI-lin-rsms})) \subseteq ((\text{B-cp-ms})).$$

*Hence, the statements (B-cp-ms), (B-cp-bdms), (E-cp-ms), and (AI-lin-rsms) are mutually equivalent.*

*Proof.* The first inclusion

$$((\text{B-cp-ms})) \subseteq ((\text{B-cp-bdms}))$$

is trivial. Further, by a simple inspection of the constructions above, (E-cp-ms) is just the principle (AI-lin-rsms) with

$$\mathcal{S} = [d < \varepsilon].$$

Hence, the third inclusion

$$((\text{E-cp-ms})) \subseteq ((\text{AI-lin-rsms}))$$

is valid. Moreover, the fourth inclusion

$$((\text{AI-lin-rsms})) \subseteq ((\text{B-cp-ms}))$$

was established in the preceding section.

It remains to prove that the second inclusion

$$((\text{B-cp-bdms})) \subseteq ((\text{E-cp-ms}))$$

is valid. Assume that the premises of (B-cp-bdms) hold:

(pre-1) $X$ is bounded,

$$\text{diam}(X) < \infty,$$

and $d$-complete;

(pre-2) $T$ is $(d; \lambda)$-contractive for some $\lambda \in [0, 1[$.

Then choose $\varepsilon > 0$ such that

$$\varepsilon > \operatorname{diam}(X).$$

Hence,

$$[d < \varepsilon] = [d < \varepsilon]^{\omega} = X \times X.$$

By this choice, it follows trivially that

(rela-1) $T$ is $(d;\lambda)$-contractive if and only if $T$ is $(d, [d < \varepsilon]; \lambda)$-contractive;

(rela-2) $T$ is $[d < \varepsilon]^{\omega}$-semi-progressive.

Thus, (E-cp-ms) is applicable to these data, and the desired inclusion follows.

Having established all the inclusions, the proof is complete. □

In particular, when

$$[d < \varepsilon]^{\omega} = X \times X,$$

that is, when $X$ is $[d < \varepsilon]^{\omega}$-connected, the Edelstein contraction principle on metric spaces (E-cp-ms) directly includes the related 1961 statement obtained by Edelstein [8]. Further aspects may be found in Turinici [29] and the references therein.

**Part-Case-2)** Let $(X, d)$ be a metric space, and let $(\le)$ be a quasi-order, that is, a reflexive and transitive relation on $X$. The triple $(X, d, \le)$ will be referred to as a quasi-ordered metric space. Denote

$$(x, y \in X): \quad x <> y \quad \text{if and only if} \quad \text{either } x \le y \text{ or } y \le x.$$

In this case, $x$ and $y$ are said to be comparable. Clearly, this relation is reflexive and symmetric, but it is not transitive in general.

The following statement, referred to as the Nieto–López linear contraction principle on quasi-ordered metric spaces (in short, (NL-lin-qoms)), is our starting point.

**Theorem 5.5.** *Let the quasi-ordered metric space $(X, d, \le)$ and the self-map $T$ of $X$ be taken as follows, under the proposed conventions:*

(i) *$T$ is $(d, <>; \lambda)$-contractive for some $\lambda \in ]0, 1[$;*

(ii) *$X$ is $(d, <>)$-complete and $(<>)$ is $d$-almost-self-closed;*

(iii) *$T$ is $(<>)$-increasing and $(<>)^{\omega}$-semi-progressive.*

*Then, $T$ is a $(<>)^{\omega}$-global strongly Picard operator modulo $(d, (<>)^{\omega})$.*

Concerning the status of this principle, the following is valid.

**Proposition 5.6.** *Under the precise context, we have*

$$((\text{B-cp-ms})) \subseteq ((\text{NL-lin-qoms})) \subseteq ((\text{AI-lin-rsms})) \subseteq ((\text{B-cp-ms})).$$

*Hence, the statements (B-cp-ms), (NL-lin-qoms), and (AI-lin-rsms) are mutually equivalent.*

*Proof.* The first inclusion

$$((\text{B-cp-ms})) \subseteq ((\text{NL-lin-qoms}))$$

is clear; it is enough to take $\le$ as $X \times X$ in (NL-lin-qoms).

Further, by a simple inspection of the constructions above, (NL-lin-qoms) is just the principle (AI-lin-rsms) with

$$\mathcal{S} = (<>).$$

Hence, the second inclusion

$$((\text{NL-lin-qoms})) \subseteq ((\text{AI-lin-rsms}))$$

follows as well.

Finally, the third inclusion

$$((\text{AI-lin-rsms})) \subseteq ((\text{B-cp-ms}))$$

was established in the preceding section. Putting these facts together, the proof is complete. □

Concerning the particular cases of our statement, it is worth noting that the $(<>)$-increasing property of the self-map $T$ is ensured by the following observation:

(Obs-1) $T$ is $(<>)$-increasing whenever it is $(\leq)$-monotone, that is, either $(\leq)$-increasing or $(\leq)$-decreasing.

In addition, the $(<>)^{\omega}$-semi-progressiveness of $T$ is ensured when

$$(<>)^{\omega} = X \times X,$$

that is, when $X$ is $(<>)^{\omega}$-connected. This holds, in particular, when

(Obs-2) $X$ is strongly $(<>)$-connected: for each $x, y \in X$, the set $\{x, y\}$ has both lower and upper bounds.

Note that, under these strong restrictions, the Nieto–López linear contraction principle on quasi-ordered metric spaces (NL-lin-qoms) is just the 2005 result of Nieto and Rodríguez-López [17]. Further technical aspects may be found in Turinici [28, Section 24] and the references therein.

## Conflict of interest

The author declares no conflict of interest.

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Submitted: April 22, 2026
Accepted: May 15, 2026
Published: May 16, 2026