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On Nonlinear Caputo Tempered Impulsive Implicit Fractional Problems

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Abstract. The main goal of this paper is to study the existence and uniqueness of impulsive implicit fractional differential equation involving the Caputo tempered (C-T) fractional derivative depending on Riemann-Liouville (R-L) tempered fractional integral. The results are based upon the Banach contraction principle, and Krasnoselskii's fixed point theorem. Furthermore, several illustrations are presented to demonstrate the plausibility of our results.

Keywords: fixed point; existence; uniqueness; tempered fractional operators; integro-differential problems
MSC (2020): 34A08, 34B10, 34B40

1 Introduction

Fractional calculus extends the concepts of differentiation and integration to non-integer orders, bridging traditional calculus with new dimensions of analysis. This innovative approach has sparked significant theoretical interest while gaining practical importance across numerous research domains. Its remarkable versatility has established fractional calculus as a cornerstone in scientific fields. Recent years have seen a notable increase in research focused on this area, exploring diverse scenarios through various forms of fractional differential equations and inclusions. For a deeper understanding of its applications, readers are encouraged to consult the works of Herrmann [8] and Samko et al. [16]. The contributions of Benchohra et al. [1, 3–5, 17] are particularly noteworthy, addressing the existence, uniqueness, and stability of solutions across a wide range of problems under distinct conditions. Notably, they introduced an extension to the Hilfer fractional derivative, which elegantly combines the Riemann-Liouville and Caputo derivatives, further enriching the field of fractional calculus.

Tempered fractional calculus has recently emerged as a prominent subset of fractional calculus, characterized by its ability to generalize various fractional operators and incorporate analytical kernels. This framework extends the scope of fractional calculus, allowing for a more comprehensive description of the continuum between regular and anomalous diffusion. The foundational definitions of fractional integration with weak singular and exponential kernels were introduced by

Buschman in [6]. For further exploration of this topic, readers are referred to [2,10–15,18]. Despite limited investigation into the Caputo tempered fractional derivative in existing literature, it holds significant potential for advancing the field. This study focuses on exploring its properties and applications within this unique mathematical framework, contributing to the ongoing development of fractional calculus.

In [10], the authors investigated the following class of Caputo tempered fractional differential equations with finite delay:

$$\begin{cases} \left({}^{C}_{0}\mathfrak{D}^{\kappa,\varepsilon}_{\delta}\mathfrak{w}\right)(\delta) = \aleph\left(\delta, \mathfrak{w}_{\delta}, \mathfrak{D}^{\kappa}_{0}\mathfrak{w}(\delta)\right), & \delta \in \Theta := [0, \varpi], \\ \mathfrak{w}(\delta) = \wp(\delta), \quad \delta \in [-\kappa, 0], \\ \jmath_1\mathfrak{w}(0) + \jmath_2\mathfrak{w}(\varpi) = \jmath_3, \end{cases}$$

where $0 < \kappa < 1$, $\varepsilon \geq 0$, ${}^{C}_{0}\mathfrak{D}^{\kappa,\varepsilon}_{\delta}$ denotes the Caputo tempered fractional derivative,

$$\aleph : \Theta \times C([-\kappa, 0], \mathbb{R}) \times \mathbb{R}$$

is a continuous function, $\wp \in C([-\kappa, \varpi], \mathbb{R})$, $0 < \varpi < +\infty$, $\jmath_1, \jmath_2, \jmath_3$ are real constants, and $\kappa > 0$ is the time delay. The results are based on the fixed point theorems of Banach, Schauder, and Schaefer. Observe that this problem encompasses initial, terminal, and anti-periodic problems; however, the employed approach does not yield solutions for the periodic problem.

In this paper, we study the existence and uniqueness of solutions for an impulsive implicit problem involving a nonlinear fractional differential equation with the Caputo tempered fractional derivative:

$$\left({}^{C}_{0}\mathfrak{D}^{\alpha,\lambda}_{t}\mathfrak{w}\right)(t) = f\left(t, \mathfrak{w}(t), {}_{0}\mathcal{I}^{\alpha,\lambda}_{t}\mathfrak{w}(t)\right), \quad t \in \mathfrak{J}_k,\ k = 0, 1, \ldots, m, \tag{1.1}$$

$$\Delta\mathfrak{w}\big|_{t=t_k} = \chi_k\left(\mathfrak{w}(t_k^-)\right), \quad k = 0, 1, \ldots, m, \tag{1.2}$$

$$\jmath_1\mathfrak{w}(0) + \jmath_2\mathfrak{w}(\varkappa) = \jmath_3, \tag{1.3}$$

where $0 < \alpha < 1$, $\lambda \geq 0$, ${}^{C}_{0}\mathfrak{D}^{\alpha,\lambda}_{t}$ and ${}_{0}\mathcal{I}^{\alpha,\lambda}_{t}$ are the Caputo tempered fractional derivative and the Riemann–Liouville tempered fractional integral, respectively, $\jmath_1, \jmath_2, \jmath_3$ are real constants, $\chi_k : \mathbb{R} \to \mathbb{R}$ are given continuous functions,

$$0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = \varkappa < \infty,$$

$$\Delta\mathfrak{w}\big|_{t=t_k} = \mathfrak{w}(t_k^+) - \mathfrak{w}(t_k^-), \quad \mathfrak{J} = [0, \varkappa], \quad \mathfrak{J}_0 = [0, t_1], \quad \mathfrak{J}_k = (t_k, t_{k+1}],\ k = 1, 2, \ldots, m,$$

and $f : \mathfrak{J} \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function.

The structure of this paper is as follows. Section 2 presents certain notations and preliminaries on the tempered fractional derivatives used throughout this manuscript. In Section 3, we present existence and uniqueness results for the problem (1.1)–(1.3), based on the fixed point theorems of Banach and Schaefer. In the last section, an illustrative example is provided in support of the obtained results.

2 Preliminaries

First, we give the definitions and notations that will be used throughout this paper. We denote by $C(\mathfrak{J}, \mathbb{R})$ the Banach space of all continuous functions from $\mathfrak{J}$ into $\mathbb{R}$, endowed with the norm

$$\|f\|_{\infty} = \sup_{t \in \mathfrak{J}} |f(t)|.$$

As usual, $AC(\mathfrak{J})$ denotes the space of absolutely continuous functions from $\mathfrak{J}$ into $\mathbb{R}$. For any $n \in \mathbb{N}^*$, we denote by $AC^n(\mathfrak{J})$ the space defined by

$$AC^n(\mathfrak{J}) := \left\{ \mathfrak{w} : \mathfrak{J} \to \mathbb{R} : \frac{d^n}{dt^n}\mathfrak{w}(t) \in AC(\mathfrak{J}) \right\}.$$

Consider the space $X_b^p(0,\varkappa)$, with $b \in \mathbb{R}$ and $1 \leq p \leq \infty$, consisting of all complex-valued Lebesgue measurable functions $\mathfrak{w}$ on $[0,\varkappa]$ such that $\|\mathfrak{w}\|_{X_b^p} < \infty$, where the norm is defined by

$$\|\mathfrak{w}\|_{X_b^p} = \left(\int_0^{\varkappa} |t^b \mathfrak{w}(t)|^p \frac{dt}{t} \right)^{\frac{1}{p}}, \quad (1 \leq p < \infty, \ b \in \mathbb{R}).$$

Consider the Banach space

$$\begin{aligned} PC(\mathfrak{J},\mathbb{R}) \ &= \ \{\mathfrak{w} : \mathfrak{J} \to \mathbb{R} : \mathfrak{w} \in C((t_k,t_{k+1}],\mathbb{R}), \ k = 0,\ldots,m, \ \text{and there exist} \\ &\quad\ \mathfrak{w}(t_k^-) \text{ and } \mathfrak{w}(t_k^+), \ k = 1,\ldots,m, \ \text{with } \mathfrak{w}(t_k^-) = \mathfrak{w}(t_k)\}, \end{aligned}$$

endowed with the norm

$$\|\mathfrak{w}\|_{PC} = \|\mathfrak{w}\|_\infty.$$

Definition 2.1 (The Riemann–Liouville tempered fractional integral [12, 15, 18]). Suppose that the real function $\mathfrak{w}$ is piecewise continuous on $[0,\varkappa]$ and $\mathfrak{w} \in X_b^p(0,\varkappa)$, with $\lambda > 0$. Then, the Riemann–Liouville tempered fractional integral of order α is defined by

$$ {}_0\mathcal{I}_t^{\alpha,\lambda}\mathfrak{w}(t) = e^{-\lambda t}\, {}_0\mathcal{I}_t^{\alpha}\left(e^{\lambda t}\mathfrak{w}(t)\right) = \frac{1}{\Gamma(\alpha)}\int_0^t \frac{e^{-\lambda(t-s)}\mathfrak{w}(s)}{(t-s)^{1-\alpha}}\,ds, \tag{2.1}$$

where ${}_0\mathcal{I}_t^{\alpha}$ denotes the Riemann–Liouville fractional integral [9], defined by

$$ {}_0\mathcal{I}_t^{\alpha}\mathfrak{w}(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \frac{\mathfrak{w}(s)}{(t-s)^{1-\alpha}}\,ds. \tag{2.2}$$

Obviously, the tempered fractional integral (2.1) reduces to the Riemann–Liouville fractional integral (2.2) when $\lambda = 0$.

Lemma 2.2 ([9], see the proof of Lemma 2.21, p. 95). *Let* $0 < \alpha < 1$. *Then, for any* $\mathfrak{w} \in C(\mathfrak{J},\mathbb{R})$, *we have*

$$\left|{}_0\mathcal{I}_t^{\alpha}\mathfrak{w}(t)\right| \leq \frac{\varkappa^\alpha}{\Gamma(\alpha+1)}\|\mathfrak{w}\|_\infty.$$

Definition 2.3 (The Riemann–Liouville tempered fractional derivative [12, 15]). For $n-1 < \alpha < n$, with $n \in \mathbb{N}^+$ and $\lambda \geq 0$, the Riemann–Liouville tempered fractional derivative is defined by

$$ {}_0\mathfrak{D}_t^{\alpha,\lambda}\mathfrak{w}(t) = e^{-\lambda t}{}_0\mathfrak{D}_t^{\alpha}\left(e^{\lambda t}\mathfrak{w}(t)\right) = \frac{e^{-\lambda t}}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t \frac{e^{\lambda s}\mathfrak{w}(s)}{(t-s)^{\alpha-n+1}}\,ds,$$

where ${}_0\mathfrak{D}_t^{\alpha}\left(e^{\lambda t}\mathfrak{w}(t)\right)$ denotes the Riemann–Liouville fractional derivative [9], given by

$$ {}_0\mathfrak{D}_t^{\alpha}\left(e^{\lambda t}\mathfrak{w}(t)\right) = \frac{d^n}{dt^n}\left({}_0\mathcal{I}_t^{n-\alpha}\left(e^{\lambda t}\mathfrak{w}(t)\right)\right) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t \frac{e^{\lambda s}\mathfrak{w}(s)}{(t-s)^{\alpha-n+1}}\,ds.$$

Definition 2.4 (The Caputo tempered fractional derivative [12, 18]). For $n-1<\alpha<n$, with $n\in\mathbb{N}^+$ and $\lambda\geq 0$, the Caputo tempered fractional derivative is defined by

$$ {}_0^C\mathfrak{D}_t^{\alpha,\lambda}\mathfrak{w}(t)=e^{-\lambda t}\,{}_0^C\mathfrak{D}_t^{\alpha}\left(e^{\lambda t}\mathfrak{w}(t)\right)=\frac{e^{-\lambda t}}{\Gamma(n-\alpha)}\int_0^t\frac{1}{(t-s)^{\alpha-n+1}}\frac{d^n\left(e^{\lambda s}\mathfrak{w}(s)\right)}{ds^n}\,ds, $$

where ${}_0^C\mathfrak{D}_t^{\alpha}\left(e^{\lambda t}\mathfrak{w}(t)\right)$ denotes the Caputo fractional derivative [9], given by

$$ {}_0^C\mathfrak{D}_t^{\alpha}\left(e^{\lambda t}\mathfrak{w}(t)\right)=\frac{1}{\Gamma(n-\alpha)}\int_0^t\frac{1}{(t-s)^{\alpha-n+1}}\frac{d^n\left(e^{\lambda s}\mathfrak{w}(s)\right)}{ds^n}\,ds. $$

Lemma 2.5 ([12]). *For a constant C, we have*

$$ {}_0\mathfrak{D}_t^{\alpha,\lambda}C=Ce^{-\lambda t}{}_0\mathfrak{D}_t^{\alpha}e^{\lambda t},\quad {}_0^C\mathfrak{D}_t^{\alpha,\lambda}C=Ce^{-\lambda t}{}_0^C\mathfrak{D}_t^{\alpha}e^{\lambda t}. $$

Obviously,

$$ {}_0\mathfrak{D}_t^{\alpha,\lambda}(C)\neq{}_0^C\mathfrak{D}_t^{\alpha,\lambda}(C). $$

Moreover, ${}_0^C\mathfrak{D}_t^{\alpha,\lambda}(C)$ is no longer equal to zero, in contrast to ${}_0^C\mathfrak{D}_t^{\alpha}(C)\equiv 0$.

Lemma 2.6 ([12, 18]). *Let $\mathfrak{w}(t)\in AC^n[0,\varkappa]$ and $n-1<\alpha<n$. Then, the Caputo tempered fractional derivative and the Riemann–Liouville tempered fractional integral satisfy the following composition properties:*

$$ {}_0\mathcal{I}_t^{\alpha,\lambda}\left[{}_0^C\mathfrak{D}_t^{\alpha,\lambda}\mathfrak{w}(t)\right]=\mathfrak{w}(t)-\sum_{k=0}^{n-1}e^{-\lambda t}\frac{(t-0)^k}{k!}\left[\frac{d^k\left(e^{\lambda t}\mathfrak{w}(t)\right)}{dt^k}\bigg|_{t=0}\right], $$

and

$$ {}_0^C\mathfrak{D}_t^{\alpha,\lambda}\left[{}_0\mathcal{I}_t^{\alpha,\lambda}\mathfrak{w}(t)\right]=\mathfrak{w}(t),\quad \text{for } \alpha\in(0,1). $$

Theorem 2.7 (Theorem of Ascoli–Arzelá [19]). *Let $A\subset PC(\mathfrak{J},\mathbb{R})$. Then A is relatively compact (i.e., $\overline{A}$ is compact) if the following conditions hold:*

1. *A is uniformly bounded, i.e., there exists $M>0$ such that*

$$ |\mathfrak{w}(t)|<M\quad \text{for every } \mathfrak{w}\in A \text{ and } t\in(t_k,t_{k+1}],\ k=1,\dots,m. $$

2. *A is equicontinuous on $(t_k,t_{k+1}]$, i.e., for every $\epsilon>0$, there exists $\delta>0$ such that for each $t,\bar{t}\in(t_k,t_{k+1}]$, the condition $\left|t-\bar{t}\right|\leq\delta$ implies*

$$ \left|\mathfrak{w}(t)-\mathfrak{w}(\bar{t})\right|\leq\epsilon,\quad \text{for every } \mathfrak{w}\in A. $$

Theorem 2.8 (Banach's fixed point theorem [7]). *Let C be a nonempty closed subset of a Banach space E. Then any contraction mapping $T:C\longrightarrow C$ has a unique fixed point.*

Theorem 2.9 (Krasnosel'skii's fixed point theorem [7]). *Let M be a nonempty, closed, and convex subset of a Banach space X, and let A,B be operators such that*

1. *$Ax+By\in M$ for all $x,y\in M$;*

2. *A is compact and continuous;*

3. *B is a contraction mapping.*

Then there exists $z\in M$ such that $z=Az+Bz$.

3 Main results

To prove the existence of solutions to (1.1)–(1.3), we need the following auxiliary lemma.

Lemma 3.1. *Let* $0 < \alpha \leq 1$*,* $\lambda \geq 0$*, and let* $\sigma : \mathfrak{J} \to \mathbb{R}$ *be continuous. Then a function* $\mathfrak{w}$ *is a solution of the fractional boundary value problem*

$$\left({}_0^C\mathfrak{D}_t^{\alpha,\lambda}\mathfrak{w}\right)(t) = \sigma(t), \quad t \in \mathfrak{J}_k, \tag{3.1}$$

$$\Delta\mathfrak{w}\Big|_{t=t_k} = \chi_k\left(\mathfrak{w}(t_k^-)\right), \quad k = 0, 1, \ldots, m, \tag{3.2}$$

$$\jmath_1\mathfrak{w}(0) + \jmath_2\mathfrak{w}(\varkappa) = \jmath_3, \tag{3.3}$$

if and only if it satisfies the fractional integral equation

$$\begin{aligned}
\mathfrak{w}(t) = {} & \frac{\jmath_3 e^{-\lambda t}}{\jmath_1 + \jmath_2 e^{-\lambda\varkappa}} - \frac{\jmath_2 e^{-\lambda t}}{\jmath_1 + \jmath_2 e^{-\lambda\varkappa}}\left[\sum_{i=1}^{m} e^{-\lambda(\varkappa - t_i)}\chi_i(\mathfrak{w}(t_i^-))\right. \\
& \left. + \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{m} e^{-\lambda(\varkappa - t_i)}\int_{t_{i-1}}^{t_i}\frac{e^{-\lambda(t_i - s)}}{(t_i - s)^{1-\alpha}}\sigma(s)\,ds + \frac{1}{\Gamma(\alpha)}\int_{t_m}^{\varkappa}\frac{e^{-\lambda(\varkappa - s)}}{(\varkappa - s)^{1-\alpha}}\sigma(s)\,ds\right] \\
& + \sum_{i=1}^{k} e^{-\lambda(t - t_i)}\chi_i(\mathfrak{w}(t_i^-)) + \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k} e^{-\lambda(t - t_i)}\int_{t_{i-1}}^{t_i}\frac{e^{-\lambda(t_i - s)}}{(t_i - s)^{1-\alpha}}\sigma(s)\,ds \\
& + \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}\frac{e^{-\lambda(t - s)}}{(t - s)^{1-\alpha}}\sigma(s)\,ds.
\end{aligned} \tag{3.4}$$

Proof. Assume that $\mathfrak{w}$ satisfies (3.1)–(3.3). If $t \in [0, t_1]$, then

$$\left({}_0^C\mathfrak{D}_t^{\alpha,\lambda}\mathfrak{w}\right)(t) = \sigma(t).$$

By Lemma 2.6, it follows that

$$\mathfrak{w}(t) = e^{-\lambda t}\mathfrak{w}(0) + {}_0\mathcal{I}_t^{\alpha,\lambda}\sigma(t) = e^{-\lambda t}\mathfrak{w}(0) + \frac{1}{\Gamma(\alpha)}\int_0^t\frac{e^{-\lambda(t - s)}}{(t - s)^{1-\alpha}}\sigma(s)\,ds.$$

If $t \in (t_1, t_2]$, then, by Lemma 2.6, we obtain

$$\begin{aligned}
\mathfrak{w}(t) &= e^{-\lambda(t - t_1)}\mathfrak{w}(t_1^+) + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t}\frac{e^{-\lambda(t - s)}}{(t - s)^{1-\alpha}}\sigma(s)\,ds \\
&= e^{-\lambda(t - t_1)}\left(\Delta\mathfrak{w}\Big|_{t=t_1} + \mathfrak{w}(t_1^-)\right) + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t}\frac{e^{-\lambda(t - s)}}{(t - s)^{1-\alpha}}\sigma(s)\,ds \\
&= e^{-\lambda(t - t_1)}\chi_1(\mathfrak{w}(t_1^-)) + e^{-\lambda(t - t_1)}\left[e^{-\lambda t_1}\mathfrak{w}(0) + \frac{1}{\Gamma(\alpha)}\int_0^{t_1}\frac{e^{-\lambda(t_1 - s)}}{(t_1 - s)^{1-\alpha}}\sigma(s)\,ds\right] \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t}\frac{e^{-\lambda(t - s)}}{(t - s)^{1-\alpha}}\sigma(s)\,ds \\
&= e^{-\lambda t}\mathfrak{w}(0) + e^{-\lambda(t - t_1)}\chi_1(\mathfrak{w}(t_1^-)) + \frac{e^{-\lambda(t - t_1)}}{\Gamma(\alpha)}\int_0^{t_1}\frac{e^{-\lambda(t_1 - s)}}{(t_1 - s)^{1-\alpha}}\sigma(s)\,ds \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t}\frac{e^{-\lambda(t - s)}}{(t - s)^{1-\alpha}}\sigma(s)\,ds.
\end{aligned}$$

If $t \in (t_2, t_3]$, then, by Lemma 2.6, we obtain

$$\begin{aligned}
\mathfrak{w}(t) &= e^{-\lambda(t-t_2)}\mathfrak{w}(t_2^+) + \frac{1}{\Gamma(\alpha)}\int_{t_2}^{t} \frac{e^{-\lambda(t-s)}}{(t-s)^{1-\alpha}}\sigma(s)\,ds \\
&= e^{-\lambda(t-t_2)}\left(\Delta\mathfrak{w}\big|_{t=t_2} + \mathfrak{w}(t_2^-)\right) + \frac{1}{\Gamma(\alpha)}\int_{t_2}^{t} \frac{e^{-\lambda(t-s)}}{(t-s)^{1-\alpha}}\sigma(s)\,ds \\
&= e^{-\lambda(t-t_2)}\chi_2(\mathfrak{w}(t_2^-)) + e^{-\lambda(t-t_2)}\Bigg[e^{-\lambda t_2}\mathfrak{w}(0) + e^{-\lambda(t_2-t_1)}\chi_1(\mathfrak{w}(t_1^-)) \\
&\quad + \frac{e^{-\lambda(t_2-t_1)}}{\Gamma(\alpha)}\int_0^{t_1} \frac{e^{-\lambda(t_1-s)}}{(t_1-s)^{1-\alpha}}\sigma(s)\,ds + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2} \frac{e^{-\lambda(t_2-s)}}{(t_2-s)^{1-\alpha}}\sigma(s)\,ds\Bigg] \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_{t_2}^{t} \frac{e^{-\lambda(t-s)}}{(t-s)^{1-\alpha}}\sigma(s)\,ds \\
&= e^{-\lambda t}\mathfrak{w}(0) + e^{-\lambda(t-t_1)}\chi_1(\mathfrak{w}(t_1^-)) + e^{-\lambda(t-t_2)}\chi_2(\mathfrak{w}(t_2^-)) + \frac{e^{-\lambda(t-t_1)}}{\Gamma(\alpha)}\int_0^{t_1} \frac{e^{-\lambda(t_1-s)}}{(t_1-s)^{1-\alpha}}\sigma(s)\,ds \\
&\quad + \frac{e^{-\lambda(t-t_2)}}{\Gamma(\alpha)}\int_{t_1}^{t_2} \frac{e^{-\lambda(t_2-s)}}{(t_2-s)^{1-\alpha}}\sigma(s)\,ds + \frac{1}{\Gamma(\alpha)}\int_{t_2}^{t} \frac{e^{-\lambda(t-s)}}{(t-s)^{1-\alpha}}\sigma(s)\,ds.
\end{aligned}$$

Repeating this process, we obtain that the solution $\mathfrak{w}(t)$, for $t \in (t_k, t_{k+1}]$ with $k = 1, \ldots, m$, can be written as

$$\begin{aligned}
\mathfrak{w}(t) &= e^{-\lambda t}\mathfrak{w}(0) + \sum_{i=1}^{k} e^{-\lambda(t-t_i)}\chi_i(\mathfrak{w}(t_i^-)) \\
&\quad + \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k} e^{-\lambda(t-t_i)}\int_{t_{i-1}}^{t_i} \frac{e^{-\lambda(t_i-s)}}{(t_i-s)^{1-\alpha}}\sigma(s)\,ds + \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t} \frac{e^{-\lambda(t-s)}}{(t-s)^{1-\alpha}}\sigma(s)\,ds.
\end{aligned}$$

Applying the boundary condition $\jmath_1\mathfrak{w}(0) + \jmath_2\mathfrak{w}(\varkappa) = \jmath_3$, we obtain

$$\begin{aligned}
\jmath_3 &= \jmath_1\mathfrak{w}(0) + \jmath_2\Bigg[e^{-\lambda\varkappa}\mathfrak{w}(0) + \sum_{i=1}^{m} e^{-\lambda(\varkappa-t_i)}\chi_i(\mathfrak{w}(t_i^-)) \\
&\quad + \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{m} e^{-\lambda(\varkappa-t_i)}\int_{t_{i-1}}^{t_i} \frac{e^{-\lambda(t_i-s)}}{(t_i-s)^{1-\alpha}}\sigma(s)\,ds + \frac{1}{\Gamma(\alpha)}\int_{t_m}^{\varkappa} \frac{e^{-\lambda(\varkappa-s)}}{(\varkappa-s)^{1-\alpha}}\sigma(s)\,ds\Bigg].
\end{aligned}$$

It follows that

$$\begin{aligned}
\mathfrak{w}(0) &= \frac{\jmath_3}{\jmath_1 + \jmath_2 e^{-\lambda\varkappa}} - \frac{\jmath_2}{\jmath_1 + \jmath_2 e^{-\lambda\varkappa}}\Bigg[\sum_{i=1}^{m} e^{-\lambda(\varkappa-t_i)}\chi_i(\mathfrak{w}(t_i^-)) \\
&\quad + \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{m} e^{-\lambda(\varkappa-t_i)}\int_{t_{i-1}}^{t_i} \frac{e^{-\lambda(t_i-s)}}{(t_i-s)^{1-\alpha}}\sigma(s)\,ds + \frac{1}{\Gamma(\alpha)}\int_{t_m}^{\varkappa} \frac{e^{-\lambda(\varkappa-s)}}{(\varkappa-s)^{1-\alpha}}\sigma(s)\,ds\Bigg].
\end{aligned}$$

Thus, for $t \in (t_k, t_{k+1}]$, where $k = 1, \ldots, m$, we have

$$\begin{aligned}\mathfrak{w}(t) = {}& \frac{\jmath_3 e^{-\lambda t}}{\jmath_1 + \jmath_2 e^{-\lambda\varkappa}} - \frac{\jmath_2 e^{-\lambda t}}{\jmath_1 + \jmath_2 e^{-\lambda\varkappa}}\Bigg[\sum_{i=1}^{m} e^{-\lambda(\varkappa - t_i)}\chi_i(\mathfrak{w}(t_i^-)) \\ & + \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{m} e^{-\lambda(\varkappa - t_i)}\int_{t_{i-1}}^{t_i}\frac{e^{-\lambda(t_i - s)}}{(t_i - s)^{1-\alpha}}\sigma(s)\,ds + \frac{1}{\Gamma(\alpha)}\int_{t_m}^{\varkappa}\frac{e^{-\lambda(\varkappa - s)}}{(\varkappa - s)^{1-\alpha}}\sigma(s)\,ds\Bigg] \\ & + \sum_{i=1}^{k} e^{-\lambda(t - t_i)}\chi_i(\mathfrak{w}(t_i^-)) + \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k} e^{-\lambda(t - t_i)}\int_{t_{i-1}}^{t_i}\frac{e^{-\lambda(t_i - s)}}{(t_i - s)^{1-\alpha}}\sigma(s)\,ds \\ & + \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}\frac{e^{-\lambda(t - s)}}{(t - s)^{1-\alpha}}\sigma(s)\,ds.\end{aligned}$$

Conversely, assume that $\mathfrak{w}$ satisfies the impulsive fractional integral equation (3.4). If $t \in [0, t_1]$, then we have

$$\begin{aligned}\mathfrak{w}(t) = {}& \frac{\jmath_3 e^{-\lambda t}}{\jmath_1 + \jmath_2 e^{-\lambda\varkappa}} - \frac{\jmath_2 e^{-\lambda t}}{\jmath_1 + \jmath_2 e^{-\lambda\varkappa}}\Bigg[\sum_{i=1}^{m} e^{-\lambda(\varkappa - t_i)}\chi_i(\mathfrak{w}(t_i^-)) \\ & + \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{m} e^{-\lambda(\varkappa - t_i)}\int_{t_{i-1}}^{t_i}\frac{e^{-\lambda(t_i - s)}}{(t_i - s)^{1-\alpha}}\sigma(s)\,ds + \frac{1}{\Gamma(\alpha)}\int_{t_m}^{\varkappa}\frac{e^{-\lambda(\varkappa - s)}}{(\varkappa - s)^{1-\alpha}}\sigma(s)\,ds\Bigg] \\ & + \frac{1}{\Gamma(\alpha)}\int_{0}^{t}\frac{e^{-\lambda(t - s)}}{(t - s)^{1-\alpha}}\sigma(s)\,ds.\end{aligned}$$

Applying ${}_0^C\mathfrak{D}_t^{\alpha,\lambda}$ to both sides, and using Lemmas 2.5 and 2.6, we obtain

$${}_0^C\mathfrak{D}_t^{\alpha,\lambda}\mathfrak{w}(t) = \sigma(t), \quad \text{for all } t \in [0, t_1].$$

If $t \in (t_k, t_{k+1}]$, with $k = 1, \ldots, m$, then

$$\begin{aligned}\mathfrak{w}(t) = {}& \frac{\jmath_3 e^{-\lambda t}}{\jmath_1 + \jmath_2 e^{-\lambda\varkappa}} - \frac{\jmath_2 e^{-\lambda t}}{\jmath_1 + \jmath_2 e^{-\lambda\varkappa}}\Bigg[\sum_{i=1}^{m} e^{-\lambda(\varkappa - t_i)}\chi_i(\mathfrak{w}(t_i^-)) \\ & + \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{m} e^{-\lambda(\varkappa - t_i)}\int_{t_{i-1}}^{t_i}\frac{e^{-\lambda(t_i - s)}}{(t_i - s)^{1-\alpha}}\sigma(s)\,ds + \frac{1}{\Gamma(\alpha)}\int_{t_m}^{\varkappa}\frac{e^{-\lambda(\varkappa - s)}}{(\varkappa - s)^{1-\alpha}}\sigma(s)\,ds\Bigg] \\ & + \sum_{i=1}^{k} e^{-\lambda(t - t_i)}\chi_i(\mathfrak{w}(t_i^-)) + \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k} e^{-\lambda(t - t_i)}\int_{t_{i-1}}^{t_i}\frac{e^{-\lambda(t_i - s)}}{(t_i - s)^{1-\alpha}}\sigma(s)\,ds \\ & + \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}\frac{e^{-\lambda(t - s)}}{(t - s)^{1-\alpha}}\sigma(s)\,ds.\end{aligned}$$

Applying ${}_0^C\mathfrak{D}_t^{\alpha,\lambda}$ to both sides, and using Lemmas 2.5 and 2.6, we obtain

$${}_0^C\mathfrak{D}_t^{\alpha,\lambda}\mathfrak{w}(t) = \sigma(t), \quad \text{for all } t \in (t_k, t_{k+1}].$$

Moreover, it is easy to verify that

$$\Delta\mathfrak{w}\big|_{t=t_k} = \chi_k(\mathfrak{w}(t_k^-)), \quad k = 1, \ldots, m,$$

and

$$\jmath_1\mathfrak{w}(0) + \jmath_2\mathfrak{w}(\varkappa) = \jmath_3.$$

This completes the proof. □

We are now in a position to state and prove our existence result for the problem (1.1)–(1.3), based on Banach's fixed point theorem.

Theorem 3.2. *Assume that:*

(H1) *There exist constants* $K, L > 0$ *such that*

$$|f(t,\mathfrak{w},\mathfrak{u}) - f(t,\overline{\mathfrak{w}},\overline{\mathfrak{u}})| \le K|\mathfrak{w}-\overline{\mathfrak{w}}| + L|\mathfrak{u}-\overline{\mathfrak{u}}|$$

for all $\mathfrak{w},\mathfrak{u},\overline{\mathfrak{w}},\overline{\mathfrak{u}} \in \mathbb{R}$ *and* $t \in \mathfrak{J}$.

(H2) *There exists a constant* $\widetilde{l} > 0$ *such that*

$$|\chi_k(\mathfrak{w}) - \chi_k(\overline{\mathfrak{w}})| \le \widetilde{l}|\mathfrak{w}-\overline{\mathfrak{w}}|,$$

for all $\mathfrak{w},\overline{\mathfrak{w}} \in \mathbb{R}$ *and* $k = 1,\dots,m$.

If

$$\left[\frac{|\jmath_2|}{|\jmath_1+\jmath_2e^{-\lambda\varkappa}|}+1\right]\left[\widetilde{l}m+\frac{Ke^{\lambda\varkappa}(m+1)\varkappa^{\alpha}}{\Gamma(\alpha+1)}+\frac{Le^{\lambda\varkappa}(m+1)\varkappa^{2\alpha}}{\Gamma(2\alpha+1)}\right]<1, \tag{3.5}$$

then the problem (1.1)–(1.3) *has a unique solution on* $\mathfrak{J}$.

Proof. We transform the problem (1.1)–(1.3) into a fixed point problem. Consider the operator

$$N : PC(\mathfrak{J},\mathbb{R}) \to PC(\mathfrak{J},\mathbb{R})$$

defined by

$$\begin{aligned} N(\mathfrak{w})(t) = {} & \frac{\jmath_3e^{-\lambda t}}{\jmath_1+\jmath_2e^{-\lambda\varkappa}} - \frac{\jmath_2e^{-\lambda t}}{\jmath_1+\jmath_2e^{-\lambda\varkappa}}\Bigg[\sum_{i=1}^{m}e^{-\lambda(\varkappa-t_i)}\chi_i(\mathfrak{w}(t_i^-)) \\ & + \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{m}e^{-\lambda(\varkappa-t_i)}\int_{t_{i-1}}^{t_i}\frac{e^{-\lambda(t_i-s)}}{(t_i-s)^{1-\alpha}}f\left(s,\mathfrak{w}(s),{}_0\mathcal{I}_s^{\alpha,\lambda}\mathfrak{w}(s)\right)ds \\ & + \frac{1}{\Gamma(\alpha)}\int_{t_m}^{\varkappa}\frac{e^{-\lambda(\varkappa-s)}}{(\varkappa-s)^{1-\alpha}}f\left(s,\mathfrak{w}(s),{}_0\mathcal{I}_s^{\alpha,\lambda}\mathfrak{w}(s)\right)ds\Bigg] \\ & + \sum_{0<t_k<t}e^{-\lambda(t-t_k)}\chi_k(\mathfrak{w}(t_k^-)) \\ & + \frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}e^{-\lambda(t-t_k)}\int_{t_{k-1}}^{t_k}\frac{e^{-\lambda(t_k-s)}}{(t_k-s)^{1-\alpha}}f\left(s,\mathfrak{w}(s),{}_0\mathcal{I}_s^{\alpha,\lambda}\mathfrak{w}(s)\right)ds \\ & + \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}\frac{e^{-\lambda(t-s)}}{(t-s)^{1-\alpha}}f\left(s,\mathfrak{w}(s),{}_0\mathcal{I}_s^{\alpha,\lambda}\mathfrak{w}(s)\right)ds. \end{aligned} \tag{3.6}$$

Clearly, the fixed points of the operator N are solutions of the problem (1.1)–(1.3).

Let $\mathfrak{w}, \mathfrak{u} \in PC(\mathfrak{J}, \mathbb{R})$. For $t \in \mathfrak{J}$, we obtain

$$
\begin{aligned}
&|N(\mathfrak{w})(t) - N(\mathfrak{u})(t)| \\
\leq\ & \frac{|\jmath_2| e^{-\lambda t}}{|\jmath_1 + \jmath_2 e^{-\lambda \varkappa}|} \Bigg[\sum_{i=1}^{m} e^{-\lambda(\varkappa - t_i)} \left|\chi_i(\mathfrak{w}(t_i^-)) - \chi_i(\mathfrak{u}(t_i^-))\right| \\
&+ \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{m} e^{-\lambda(\varkappa - t_i)} \int_{t_{i-1}}^{t_i} \frac{e^{-\lambda(t_i - s)}}{(t_i - s)^{1-\alpha}} \left| f\left(s, \mathfrak{w}(s), {}_0\mathcal{I}_s^{\alpha,\lambda}\mathfrak{w}(s)\right) - f\left(s, \mathfrak{u}(s), {}_0\mathcal{I}_s^{\alpha,\lambda}\mathfrak{u}(s)\right) \right| ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_m}^{\varkappa} \frac{e^{-\lambda(\varkappa - s)}}{(\varkappa - s)^{1-\alpha}} \left| f\left(s, \mathfrak{w}(s), {}_0\mathcal{I}_s^{\alpha,\lambda}\mathfrak{w}(s)\right) - f\left(s, \mathfrak{u}(s), {}_0\mathcal{I}_s^{\alpha,\lambda}\mathfrak{u}(s)\right) \right| ds \Bigg] \\
&+ \sum_{0<t_k<t} e^{-\lambda(t - t_k)} \left|\chi_k(\mathfrak{w}(t_k^-)) - \chi_k(\mathfrak{u}(t_k^-))\right| \\
&+ \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} e^{-\lambda(t - t_k)} \int_{t_{k-1}}^{t_k} \frac{e^{-\lambda(t_k - s)}}{(t_k - s)^{1-\alpha}} \left| f\left(s, \mathfrak{w}(s), {}_0\mathcal{I}_s^{\alpha,\lambda}\mathfrak{w}(s)\right) - f\left(s, \mathfrak{u}(s), {}_0\mathcal{I}_s^{\alpha,\lambda}\mathfrak{u}(s)\right) \right| ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} \frac{e^{-\lambda(t - s)}}{(t - s)^{1-\alpha}} \left| f\left(s, \mathfrak{w}(s), {}_0\mathcal{I}_s^{\alpha,\lambda}\mathfrak{w}(s)\right) - f\left(s, \mathfrak{u}(s), {}_0\mathcal{I}_s^{\alpha,\lambda}\mathfrak{u}(s)\right) \right| ds.
\end{aligned}
$$

Next, using Lemma 2.2, (H1), and (H2), we obtain

$$
\begin{aligned}
&|N(\mathfrak{w})(t) - N(\mathfrak{u})(t)| \\
\leq\ & \frac{|\jmath_2|}{|\jmath_1 + \jmath_2 e^{-\lambda \varkappa}|} \Bigg[\sum_{i=1}^{m} \widetilde{l} \left|\mathfrak{w}(t_i^-) - \mathfrak{u}(t_i^-)\right| \\
&+ \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{m} \int_0^{t_i} \frac{e^{-\lambda(t_i - s)}}{(t_i - s)^{1-\alpha}} \left[K\left|\mathfrak{w}(s) - \mathfrak{u}(s)\right| + L\left|{}_0\mathcal{I}_s^{\alpha,\lambda}\mathfrak{w}(s) - {}_0\mathcal{I}_s^{\alpha,\lambda}\mathfrak{u}(s)\right| \right] ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^{\varkappa} \frac{e^{-\lambda(\varkappa - s)}}{(\varkappa - s)^{1-\alpha}} \left[K\left|\mathfrak{w}(s) - \mathfrak{u}(s)\right| + L\left|{}_0\mathcal{I}_s^{\alpha,\lambda}\mathfrak{w}(s) - {}_0\mathcal{I}_s^{\alpha,\lambda}\mathfrak{u}(s)\right| ds \right] \Bigg] + \sum_{k=1}^{m} \widetilde{l} \left|\mathfrak{w}(t_k^-) - \mathfrak{u}(t_k^-)\right| \\
&+ \frac{1}{\Gamma(\alpha)} \sum_{k=1}^{m} \int_0^{t_k} \frac{e^{-\lambda(t_k - s)}}{(t_k - s)^{1-\alpha}} \left[K\left|\mathfrak{w}(s) - \mathfrak{u}(s)\right| + L\left|{}_0\mathcal{I}_s^{\alpha,\lambda}\mathfrak{w}(s) - {}_0\mathcal{I}_s^{\alpha,\lambda}\mathfrak{u}(s)\right| \right] ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^{t} \frac{e^{-\lambda(t - s)}}{(t - s)^{1-\alpha}} \left[K\left|\mathfrak{w}(s) - \mathfrak{u}(s)\right| + L\left|{}_0\mathcal{I}_s^{\alpha,\lambda}\mathfrak{w}(s) - {}_0\mathcal{I}_s^{\alpha,\lambda}\mathfrak{u}(s)\right| \right] ds \\
\leq\ & \frac{|\jmath_2|}{|\jmath_1 + \jmath_2 e^{-\lambda \varkappa}|} \Bigg[\sum_{i=1}^{m} \widetilde{l} \left|\mathfrak{w}(t_i^-) - \mathfrak{u}(t_i^-)\right| \\
&+ \frac{K}{\Gamma(\alpha)} \sum_{i=1}^{m} \int_0^{t_i} (t_i - s)^{\alpha-1} \left|\mathfrak{w}(s) - \mathfrak{u}(s)\right| ds + L \sum_{i=1}^{m} \left({}_0\mathcal{I}_{t_i}^{2\alpha,\lambda} \left|(\mathfrak{w} - \mathfrak{u})(s)\right| \right)(t_i) \Bigg] ds \\
&+ \frac{K}{\Gamma(\alpha)} \int_0^{\varkappa} (\varkappa - s)^{\alpha-1} \left|\mathfrak{w}(s) - \mathfrak{u}(s)\right| ds + L \left({}_0\mathcal{I}_{\varkappa}^{2\alpha,\lambda} \left|(\mathfrak{w} - \mathfrak{u})(s)\right| \right)(\varkappa) \Bigg] \\
&+ \sum_{k=1}^{m} \widetilde{l} \left|\mathfrak{w}(t_k^-) - \mathfrak{u}(t_k^-)\right| + \frac{K}{\Gamma(\alpha)} \sum_{k=1}^{m} \int_0^{t_k} (t_k - s)^{\alpha-1} \left|\mathfrak{w}(s) - \mathfrak{u}(s)\right| ds + L \sum_{k=1}^{m} \left({}_0\mathcal{I}_{t_k}^{2\alpha,\lambda} \left|(\mathfrak{w} - \mathfrak{u})(s)\right| \right)(t_k) \\
&+ \frac{K}{\Gamma(\alpha)} \int_0^{t} (t - s)^{\alpha-1} \left|\mathfrak{w}(s) - \mathfrak{u}(s)\right| ds + L \left({}_0\mathcal{I}_{t}^{2\alpha,\lambda} \left|(\mathfrak{w} - \mathfrak{u})(s)\right| \right)(t) \\
\leq\ & \left[\frac{|\jmath_2|}{|\jmath_1 + \jmath_2 e^{-\lambda \varkappa}|} + 1 \right] \left[\widetilde{l} m + \frac{K e^{\lambda \varkappa}(m+1)\varkappa^{\alpha}}{\Gamma(\alpha+1)} + \frac{L e^{\lambda \varkappa}(m+1)\varkappa^{2\alpha}}{\Gamma(2\alpha+1)} \right] \|\mathfrak{w} - \mathfrak{u}\|_{PC}.
\end{aligned}
$$

Thus,

$$\|N(\mathfrak{w})-N(\mathfrak{u})\|_{PC} \leq \left[\frac{|\jmath_2|}{|\jmath_1+\jmath_2 e^{-\lambda\varkappa}|}+1\right]\left[\tilde{l}m+\frac{Ke^{\lambda\varkappa}(m+1)\varkappa^{\alpha}}{\Gamma(\alpha+1)}+\frac{Le^{\lambda\varkappa}(m+1)\varkappa^{2\alpha}}{\Gamma(2\alpha+1)}\right]\|\mathfrak{w}-\mathfrak{u}\|_{PC}.$$

By (3.5), the operator N is a contraction. Hence, by Banach's contraction principle, N admits a unique fixed point, which is the unique solution of the problem (1.1)–(1.3). □

Our second result is based on Krasnoselskii's fixed point theorem.

Theorem 3.3. *Assume that* $(H1)$ *and* $(H2)$ *hold. If*

$$\left[\frac{|\jmath_2|}{|\jmath_1+\jmath_2 e^{-\lambda\varkappa}|}+1\right]\left[\tilde{l}m+\frac{K(m+1)\varkappa^{\alpha}}{\Gamma(\alpha+1)}+\frac{Lm\varkappa^{2\alpha}}{\Gamma(2\alpha+1)}\right]<1,$$

then the problem (1.1)–(1.3) *has at least one solution.*

Proof. Consider the set

$$B_{\eta^*}=\{y\in PC(\mathfrak{J},\mathbb{R}):\|y\|_{PC}\leq\eta^*\},$$

where

$$\eta^*\geq\frac{\frac{|\jmath_3|}{|\jmath_1+\jmath_2 e^{-\lambda\varkappa}|}+\left[\frac{|\jmath_2|}{|\jmath_1+\jmath_2 e^{-\lambda\varkappa}|}+1\right]\left[m\chi^*+\frac{f^*(m+1)\varkappa^{\alpha}}{\Gamma(\alpha+1)}\right]}{1-\left[\frac{|\jmath_2|}{|\jmath_1+\jmath_2 e^{-\lambda\varkappa}|}+1\right]\left[\tilde{l}m+\frac{K(m+1)\varkappa^{\alpha}}{\Gamma(\alpha+1)}+\frac{L(m+1)\varkappa^{2\alpha}}{\Gamma(2\alpha+1)}\right]},\tag{3.7}$$

with

$$f^*=\sup_{t\in\mathfrak{J}}|f(t,0,0)|,\qquad \chi^*=\max_{1\leq i\leq m}|\chi_i|.$$

We define the operators P and Q on B_{η^*} by

$$\begin{aligned}P\mathfrak{w}(t) \;=\;& \frac{\jmath_3 e^{-\lambda t}}{\jmath_1+\jmath_2 e^{-\lambda\varkappa}}-\frac{\jmath_2 e^{-\lambda t}}{\jmath_1+\jmath_2 e^{-\lambda\varkappa}}\Bigg[\sum_{i=1}^{m}e^{-\lambda(\varkappa-t_i)}\chi_i(\mathfrak{w}(t_i^-))\\
&+\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{m}e^{-\lambda(\varkappa-t_i)}\int_{t_{i-1}}^{t_i}\frac{e^{-\lambda(t_i-s)}}{(t_i-s)^{1-\alpha}}f\left(s,\mathfrak{w}(s),{}_0\mathcal{I}_s^{\alpha,\lambda}\mathfrak{w}(s)\right)ds\\
&+\frac{1}{\Gamma(\alpha)}\int_{t_m}^{\varkappa}\frac{e^{-\lambda(\varkappa-s)}}{(\varkappa-s)^{1-\alpha}}f\left(s,\mathfrak{w}(s),{}_0\mathcal{I}_s^{\alpha,\lambda}\mathfrak{w}(s)\right)ds\Bigg]+\sum_{0<t_k<t}e^{-\lambda(t-t_k)}\chi_k(\mathfrak{w}(t_k^-))\\
&+\frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}e^{-\lambda(t-t_k)}\int_{t_{k-1}}^{t_k}\frac{e^{-\lambda(t_k-s)}}{(t_k-s)^{1-\alpha}}f\left(s,\mathfrak{w}(s),{}_0\mathcal{I}_s^{\alpha,\lambda}\mathfrak{w}(s)\right)ds,\quad t\in\mathfrak{J},\end{aligned}\tag{3.8}$$

and

$$Q\mathfrak{w}(t)=\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}\frac{e^{-\lambda(t-s)}}{(t-s)^{1-\alpha}}f\left(s,\mathfrak{w}(s),{}_0\mathcal{I}_s^{\alpha,\lambda}\mathfrak{w}(s)\right)ds,\quad t\in\mathfrak{J}.\tag{3.9}$$

Then, the fractional integral equation (3.6) can be written as the operator equation

$$N\mathfrak{w}(t)=P\mathfrak{w}(t)+Q\mathfrak{w}(t),\quad \text{for } \mathfrak{w}\in PC(\mathfrak{J},\mathbb{R}).$$

The proof will be given in several steps.

Step 1: We prove that $P\mathfrak{w} + Q\mathfrak{u} \in B_{\eta^*}$ for any $\mathfrak{w}, \mathfrak{u} \in B_{\eta^*}$. Let $\mathfrak{w} \in PC(\mathfrak{J}, \mathbb{R})$. Then, for $t \in \mathfrak{J}$, we have

$$
\begin{aligned}
|P\mathfrak{w}(t)| \leq\ & \frac{|\jmath_3|}{|\jmath_1 + \jmath_2 e^{-\lambda\varkappa}|} + \frac{|\jmath_2|}{|\jmath_1 + \jmath_2 e^{-\lambda\varkappa}|}\Bigg[\sum_{i=1}^{m}\left|\chi_i(\mathfrak{w}(t_i^-)) - \chi_i(0)\right| + \sum_{i=1}^{m}|\chi_i(0)| \\
& + \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{m}\int_0^{t_i}\frac{e^{-\lambda(t_i-s)}}{(t_i-s)^{1-\alpha}}\left|f\left(s, \mathfrak{w}(s), {}_0\mathcal{I}_s^{\alpha,\lambda}\mathfrak{w}(s)\right) - f(s,0,0)\right| ds \\
& + \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{m}\int_0^{t_i}\frac{e^{-\lambda(t_i-s)}}{(t_i-s)^{1-\alpha}}|f(s,0,0)|\, ds \\
& + \frac{1}{\Gamma(\alpha)}\int_0^{\varkappa}\frac{e^{-\lambda(\varkappa-s)}}{(\varkappa-s)^{1-\alpha}}\left|f\left(s, \mathfrak{w}(s), {}_0\mathcal{I}_s^{\alpha,\lambda}\mathfrak{w}(s)\right) - f(s,0,0)\right| ds \\
& + \frac{1}{\Gamma(\alpha)}\int_0^{\varkappa}\frac{e^{-\lambda(\varkappa-s)}}{(\varkappa-s)^{1-\alpha}}|f(s,0,0)|\, ds\Bigg] \\
& + \sum_{k=1}^{m}\left|\chi_k(\mathfrak{w}(t_k^-)) - \chi_k(0)\right| + \sum_{k=1}^{m}|\chi_k(0)| \\
& + \frac{1}{\Gamma(\alpha)}\sum_{k=1}^{m}\int_0^{t_k}\frac{e^{-\lambda(t_k-s)}}{(t_k-s)^{1-\alpha}}\left|f\left(s, \mathfrak{w}(s), {}_0\mathcal{I}_s^{\alpha,\lambda}\mathfrak{w}(s)\right) - f(s,0,0)\right| ds \\
& + \frac{1}{\Gamma(\alpha)}\sum_{k=1}^{m}\int_0^{t_k}\frac{e^{-\lambda(t_k-s)}}{(t_k-s)^{1-\alpha}}|f(s,0,0)|\, ds \\
\leq\ & \frac{|\jmath_3|}{|\jmath_1 + \jmath_2 e^{-\lambda\varkappa}|} + \frac{|\jmath_2|}{|\jmath_1 + \jmath_2 e^{-\lambda\varkappa}|}\Bigg[m\chi^* + \frac{f^*(m+1)\varkappa^{\alpha}}{\Gamma(\alpha+1)} + \widetilde{l}m\eta^* \\
& + \frac{\eta^* K(m+1)\varkappa^{\alpha}}{\Gamma(\alpha+1)} + \frac{\eta^* L(m+1)\varkappa^{2\alpha}}{\Gamma(2\alpha+1)}\Bigg] \\
& + \left[\widetilde{l}m\eta^* + m\chi^* + \frac{\eta^* K m\varkappa^{\alpha}}{\Gamma(\alpha+1)} + \frac{\eta^* L m\varkappa^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{f^*(m+1)\varkappa^{\alpha}}{\Gamma(\alpha+1)}\right].
\end{aligned}
$$

This gives

$$
\begin{aligned}
\|P\mathfrak{w}\|_{PC} \leq\ & \frac{|\jmath_3|}{|\jmath_1 + \jmath_2 e^{-\lambda\varkappa}|} + \frac{|\jmath_2|}{|\jmath_1 + \jmath_2 e^{-\lambda\varkappa}|}\Bigg[m\chi^* + \frac{f^*(m+1)\varkappa^{\alpha}}{\Gamma(\alpha+1)} + \widetilde{l}m\eta^* \\
& + \frac{\eta^* K(m+1)\varkappa^{\alpha}}{\Gamma(\alpha+1)} + \frac{\eta^* L(m+1)\varkappa^{2\alpha}}{\Gamma(2\alpha+1)}\Bigg] \\
& + \left[\widetilde{l}m\eta^* + m\chi^* + \frac{\eta^* K m\varkappa^{\alpha}}{\Gamma(\alpha+1)} + \frac{\eta^* L m\varkappa^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{f^*(m+1)\varkappa^{\alpha}}{\Gamma(\alpha+1)}\right]. \qquad (3.10)
\end{aligned}
$$

Thus, (3.9) implies

$$
\begin{aligned}
|Q(\mathfrak{u})(t)| \leq\ & \frac{1}{\Gamma(\alpha)}\int_0^{t}\frac{e^{-\lambda(t-s)}}{(t-s)^{1-\alpha}}\left|f\left(s, \mathfrak{u}(s), {}_0\mathcal{I}_s^{\alpha,\lambda}\mathfrak{u}(s)\right) - f(s,0,0)\right| ds \\
& + \frac{1}{\Gamma(\alpha)}\int_0^{t}\frac{e^{-\lambda(t-s)}}{(t-s)^{1-\alpha}}|f(s,0,0)|\, ds \\
\leq\ & \frac{f^*\varkappa^{\alpha}}{\Gamma(\alpha+1)} + \frac{K\varkappa^{\alpha}\eta^*}{\Gamma(\alpha+1)} + \frac{L\varkappa^{2\alpha}\eta^*}{\Gamma(2\alpha+1)}.
\end{aligned}
$$

Therefore,

$$\|Q\mathfrak{u}\|_{PC} \leq \frac{f^*\varkappa^\alpha}{\Gamma(\alpha+1)} + \frac{K\varkappa^\alpha\eta^*}{\Gamma(\alpha+1)} + \frac{L\varkappa^{2\alpha}\eta^*}{\Gamma(2\alpha+1)}. \tag{3.11}$$

Combining (3.10) and (3.11), for every $\mathfrak{w}, \mathfrak{u} \in B_{\eta^*}$, we obtain

$$\begin{aligned} \|P\mathfrak{w} + Q\mathfrak{u}\|_{PC} &\leq \|P\mathfrak{w}\|_{PC} + \|Q\mathfrak{u}\|_{PC} \\ &\leq \frac{|\jmath_3|}{|\jmath_1 + \jmath_2 e^{-\lambda\varkappa}|} + \left[\frac{|\jmath_2|}{|\jmath_1 + \jmath_2 e^{-\lambda\varkappa}|} + 1\right]\left[m\chi^* + \frac{f^*(m+1)\varkappa^\alpha}{\Gamma(\alpha+1)}\right] \\ &\quad + \left[\frac{|\jmath_2|}{|\jmath_1 + \jmath_2 e^{-\lambda\varkappa}|} + 1\right]\left[\widetilde{l}m + \frac{K(m+1)\varkappa^\alpha}{\Gamma(\alpha+1)} + \frac{L(m+1)\varkappa^{2\alpha}}{\Gamma(2\alpha+1)}\right]\eta^*. \end{aligned}$$

Then, it follows from (3.7) that

$$\|P\mathfrak{w} + Q\mathfrak{u}\|_{PC} \leq \eta^*,$$

which implies that $P\mathfrak{w} + Q\mathfrak{u} \in B_{\eta^*}$.

Step 2: Clearly, P is a contraction.

Step 3: Q is compact and continuous.
The continuity of Q follows from the continuity of f. Next, we prove that Q is uniformly bounded on B_{η^*}. Let $\mathfrak{w} \in B_{\eta^*}$. Then, by (3.11), we have

$$\|Q\mathfrak{w}\|_{PC} \leq \frac{f^*\varkappa^\alpha}{\Gamma(\alpha+1)} + \frac{K\varkappa^\alpha\eta^*}{\Gamma(\alpha+1)} + \frac{L\varkappa^{2\alpha}\eta^*}{\Gamma(2\alpha+1)}.$$

This shows that Q is uniformly bounded on B_{η^*}.

Next, we show that QB_{η^*} is equicontinuous. Let $\mathfrak{w} \in B_{\eta^*}$ and let $0 < \delta_1 < \delta_2 \leq \varkappa$. Then,

$$\begin{aligned} &|Q\mathfrak{w}(\delta_1) - Q\mathfrak{w}(\delta_2)| \\ \leqslant\ & \frac{1}{\Gamma(\alpha)}\int_0^{\delta_1}\left|(\delta_2-s)^{\alpha-1}e^{-\lambda(\delta_2-s)} - (\delta_1-s)^{\alpha-1}e^{-\lambda(\delta_1-s)}\right|\left|f(s,\mathfrak{w}(s),{}_0\mathcal{I}_s^{\alpha,\lambda}\mathfrak{w}(s))\right|ds \\ & + \frac{1}{\Gamma(\alpha)}\int_{\delta_1}^{\delta_2}(\delta_2-s)^{\alpha-1}e^{-\lambda(\delta_2-s)}\left|f(s,\mathfrak{w}(s),{}_0\mathcal{I}_s^{\alpha,\lambda}\mathfrak{w}(s))\right|ds \\ \leqslant\ & \frac{1}{\Gamma(\alpha)}\int_0^{\delta_1}\left|(\delta_2-s)^{\alpha-1}e^{-\lambda(\delta_2-s)} - (\delta_1-s)^{\alpha-1}e^{-\lambda(\delta_1-s)}\right|\left|f(s,\mathfrak{w}(s),{}_0\mathcal{I}_s^{\alpha,\lambda}\mathfrak{w}(s)) - f(s,0,0)\right|ds \\ & + \frac{1}{\Gamma(\alpha)}\int_0^{\delta_1}\left|(\delta_2-s)^{\alpha-1}e^{-\lambda(\delta_2-s)} - (\delta_1-s)^{\alpha-1}e^{-\lambda(\delta_1-s)}\right||f(s,0,0)|\,ds \\ & + \frac{1}{\Gamma(\alpha)}\int_{\delta_1}^{\delta_2}(\delta_2-s)^{\alpha-1}e^{-\lambda(\delta_2-s)}\left|f(s,\mathfrak{w}(s),{}_0\mathcal{I}_s^{\alpha,\lambda}\mathfrak{w}(s)) - f(s,0,0)\right|ds \\ & + \frac{1}{\Gamma(\alpha)}\int_{\delta_1}^{\delta_2}(\delta_2-s)^{\alpha-1}e^{-\lambda(\delta_2-s)}|f(s,0,0)|\,ds \\ \leqslant\ & \frac{K\eta^* + f^*}{\Gamma(\alpha)}\int_0^{\delta_1}\left|(\delta_2-s)^{\alpha-1}e^{-\lambda(\delta_2-s)} - (\delta_1-s)^{\alpha-1}e^{-\lambda(\delta_1-s)}\right|ds \\ & + \frac{L\eta^*}{\Gamma(2\alpha)}\left[\int_0^{\delta_1}(\delta_1-s)^{2\alpha-1}e^{-\lambda(\delta_1-s)}\,ds - \int_0^{\delta_2}(\delta_2-s)^{2\alpha-1}e^{-\lambda(\delta_2-s)}\,ds\right] \\ & + \left[\frac{2L\eta^*\delta_2^\alpha}{\Gamma(\alpha)\Gamma(\alpha+1)} + \frac{2K\eta^* + f^*}{\Gamma(\alpha)}\right]\int_{\delta_1}^{\delta_2}(\delta_2-s)^{\alpha-1}e^{-\lambda(\delta_2-s)}\,ds. \end{aligned}$$

Note that

$$|Q\mathfrak{u}(\delta_1) - Q\mathfrak{u}(\delta_2)| \to 0 \quad \text{as } \delta_2 \to \delta_1.$$

This shows that QB_{η^*} is equicontinuous on $\mathfrak{J}$. Therefore, QB_{η^*} is relatively compact on B_{η^*}. By the Arzelá–Ascoli theorem, Q is compact on B_{η^*}.

By Krasnoselskii's fixed point theorem, the operator N has at least one fixed point, which implies that the problem (1.1)–(1.3) has at least one solution. □

4 Example

Consider the following boundary value problem with impulse:

$$
{}_0^C\mathfrak{D}_t^{\frac{1}{2},2}\mathfrak{w}(t) = f\left(t, \mathfrak{w}(t), {}_0\mathcal{I}_t^{\frac{1}{2},1}\mathfrak{w}(t)\right), \quad t \in J_0 \cup J_1, \tag{4.1}
$$

$$
\Delta\mathfrak{w}|_{t=\frac{1}{2}} = \chi_1\left(\mathfrak{w}\left(\frac{1}{2}\right)\right), \tag{4.2}
$$

$$
\frac{2}{3}\mathfrak{w}(0) + \frac{6}{7}\mathfrak{w}(1) = \frac{1}{3}, \tag{4.3}
$$

where $J_0 = \left[0, \frac{1}{2}\right]$ and $J_1 = \left(\frac{1}{2}, 1\right]$.

We have

$$
f\left(t, \mathfrak{w}(t), {}_0\mathcal{I}_t^{\frac{1}{2},1}\mathfrak{w}(t)\right) = \frac{3\ln^{\frac{-3}{7}}(t+1)}{e^{5t}+2} + \frac{t}{e^3\sqrt{\pi}}\mathfrak{w}(t) + \frac{e^{-7-t^2}}{33e^{11}}\left(\frac{{}_0\mathcal{I}_t^{\frac{1}{2},2}\mathfrak{w}(t)}{1 + {}_0\mathcal{I}_t^{\frac{1}{2},2}\mathfrak{w}(t)}\right),
$$

and

$$
\chi_1\left(\mathfrak{w}\left(\frac{1}{2}\right)\right) = \frac{\mathfrak{w}\left(\frac{1}{2}\right)}{5 + \mathfrak{w}\left(\frac{1}{2}\right)}.
$$

Here $\alpha = \frac{1}{2}$, $\lambda = 1$, $\jmath_1 = \frac{2}{3}$, $\jmath_2 = \frac{6}{7}$, $\jmath_3 = \frac{1}{3}$, and $\varkappa = 1$.

It is clear that $f \in C([0,1], \mathbb{R})$.

Let $\mathfrak{w}, \overline{\mathfrak{w}}, \upsilon, \overline{\upsilon} \in \mathbb{R}$ and $t \in \mathfrak{J}$. Then

$$
\left|f(t, \mathfrak{w}, \upsilon) - f\left(t, \overline{\mathfrak{w}}, \overline{\upsilon}\right)\right| \leqslant \frac{1}{e^3\sqrt{\pi}}|\mathfrak{w} - \overline{\mathfrak{w}}| + \frac{1}{33e^{19}}|\upsilon - \overline{\upsilon}|.
$$

Moreover,

$$
\left|\chi_1(\mathfrak{w}) - \chi_1(\overline{\mathfrak{w}})\right| \leqslant \frac{1}{5}|\mathfrak{w} - \overline{\mathfrak{w}}|.
$$

Hence, conditions $(H1)$ and $(H2)$ are satisfied with

$$
K = \frac{1}{e^3\sqrt{\pi}}, \quad L = \frac{1}{33e^{19}}, \quad \widetilde{l} = \frac{1}{5}.
$$

The condition

$$
\left[\frac{|\jmath_2|}{|\jmath_1 + \jmath_2 e^{-\lambda\varkappa}|} + 1\right]\left[\widetilde{l}m + \frac{eK(m+1)\varkappa^{\alpha}}{\Gamma(\alpha+1)} + \frac{eL(m+1)\varkappa^{2\alpha}}{\Gamma(2\alpha+1)}\right] \approx 0.78005 < 1
$$

is satisfied. It follows from Theorem 3.2 that the problem (4.1)–(4.3) has a unique solution.

Conflict of interest

The authors declare that there are no conflicts of interest.

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