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A Nonlocal Problem for the Time-Fractional Diffusion-Wave Equation

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Abstract. In this paper, we study a nonlocal initial boundary value problem for an abstract time-fractional diffusion-wave equation involving an unbounded, positive, self-adjoint operator on a Hilbert space. The existence of a mild solution is investigated using the eigenfunction decomposition method. By expanding the solution in terms of the operators eigenfunctions, the problem is reduced to a system of ordinary fractional differential equations with nonlocal conditions. The solutions of these equations are expressed in terms of the Mittag-Leffler function. By examining the zeros of the denominator, we identify the appropriate interval of definition, which excludes the right endpoint, ensuring the correctness of the solution. The solution to the original problem is expressed as a series expansion in terms of the eigenfunctions of an abstract operator. By applying estimation techniques in the corresponding Hilbert spaces, we establish the existence, uniqueness, and regularity of the solution.

Keywords: fractional diffusion-wave equation; Mittag-Leffler function; Hölder continuity; existence; uniqueness; regularity
MSC (2020): 35A09, 35R11, 35R30, 45B05

1 Introduction

Let $T > 0$ be a fixed constant, and let H be a separable Hilbert space equipped with the inner product $(\cdot, \cdot)$ and the associated norm $\|\cdot\|$. Let

$$A : D(A) \subset H \to H$$

be an unbounded, positive, self-adjoint operator on H such that its inverse A^{-1} exists and is a compact operator on H. Moreover, $0 \notin \sigma(A)$.

Suppose that A has a complete system of orthonormal eigenfunctions $\{e_n\}$ in H, associated with a countable set of positive eigenvalues $\{\lambda_n\}$. It is convenient to assume that the eigenvalues are nondecreasing, that is,

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \to +\infty.$$

We consider the following fractional wave equation:

$$\partial_t^\alpha u(t) + Au(t) = F(t), \quad t \in (0, T), \tag{1.1}$$

where ∂_t^α denotes the Caputo time-fractional derivative of order $1 < \alpha < 2$, defined by

$$\partial_t^\alpha y(t) = \frac{t_+^{1-\alpha}}{\Gamma(2-\alpha)} \star y''(t),$$

where $\star$ denotes the convolution, $\Gamma(\cdot)$ is Euler's Gamma function, and

$$t_+^{1-\alpha} = \begin{cases} t^{1-\alpha}, & t > 0, \\ 0, & t \le 0, \end{cases}$$

and the source term $F(t)$ is given.

We supplement equation (1.1) with the following non-local initial conditions:

$$\gamma u(0) + u(T) = \varphi, \quad u'(T) = \psi, \tag{1.2}$$

where $\gamma \in (-\infty, -1) \cup (0, \infty)$ is a given number, and φ and ψ are given elements of H.

Definition 1.1. A function $u(t)$, represented by the series

$$u(t) = \sum_{n=1}^{\infty} u_n(t)\, e_n,$$

is called a mild solution of the fractional problem (1.1)–(1.2) if

$$u \in C([0,T]; D(A)) \cap C^1([0,T]; H),$$

and its Fourier coefficients $u_n(t) = (u(t), e_n)$ satisfy the scalar fractional differential equations

$$\partial_t^\alpha u_n(t) + \lambda_n u_n(t) = F_n(t), \quad t \in (0,T), \quad 1 < \alpha < 2,$$

subject to the nonlocal conditions

$$\gamma u_n(0) + u_n(T) = \varphi_n, \qquad u_n'(T) = \psi_n, \qquad n \in \mathbb{N},$$

where

$$\varphi_n = (\varphi, e_n), \qquad \psi_n = (\psi, e_n), \qquad F_n(t) = (F(t), e_n).$$

The fractional diffusion-wave (or superdiffusion) equation with Caputo time derivatives of order $1 < \alpha < 2$ has attracted substantial interest due to its ability to model phenomena exhibiting both diffusive and wave-like characteristics. Foundational work by Sakamoto and Yamamoto [20] rigorously established the well-posedness and regularity of solutions to initial and boundary value problems for such equations, along with important results on related inverse problems, including source and coefficient identification.

Building on this foundation, recent studies have extended the analysis to fractional partial differential equations with nonlocal or multi-point initial conditions and have explored backward and ill-posed inverse problems. For example, Alimov and Ashurov [1] examined backward problems for time-fractional subdiffusion equations, developing stability estimates and regularization methods that are essential for handling the inherent ill-posedness of such problems. Similarly, Ashurov and Fayziev [2] studied the fractional diffusion equation with nonlocal initial conditions, providing existence, uniqueness, and stability results for direct and inverse source problems within a spectral framework.

Moreover, Floridia, Li, and Yamamoto [8] analyzed the well-posedness of backward problems in time for general time-fractional diffusion equations, contributing important theoretical insights into stability and uniqueness. Nonlocal-in-time conditions arise naturally in applications where control or observation is performed at the terminal time or over time intervals, and they have been rigorously investigated in the recent literature.

In particular, Ruzhansky et al. [19] developed a detailed spectral framework for fractional diffusion-wave equations under multi-point and integral conditions, proving existence and uniqueness via Mittag-Leffler expansions. Their approach is particularly well suited to Hilbert space settings, where the operator A admits a complete set of eigenfunctions.

These developments motivate the present study, in which we consider fractional diffusion-wave equations in a separable Hilbert space equipped with a positive self-adjoint operator, together with nonlocal time conditions.

In recent years, alongside classical conditional problems, inverse problems involving nonlocal conditions have been extensively investigated. In particular, Kirane, Malik, and Al-Gwaiz [11] addressed such an inverse problem for a two-dimensional time-fractional diffusion equation with nonlocal boundary conditions. Specifically, the authors considered a linear time-fractional diffusion equation defined on a rectangular spatial domain, involving a Caputo time-fractional derivative and nonlocal boundary conditions. The main objective was to determine an unknown source term, independent of time but possibly depending on the spatial variables, from additional integral-type measurements and nonlocal boundary specifications.

Xing Cheng and Zhiyuan Li [21] investigated an inverse source problem for a time-fractional diffusion-wave equation in a bounded domain, focusing on the recovery of a spatially dependent source term from additional measurement data provided on part of the boundary. Lingyun Qiu and Jiwoon Sim [13] studied the uniqueness of recovering either the spatial or the temporal component of a separated source term in a time-fractional diffusion-wave equation from limited observation data.

In the context of fractional diffusion-wave equations, which interpolate between parabolic and hyperbolic behavior through a Caputo time-fractional derivative of order $\alpha \in (1, 2)$, ill-posedness typically manifests as severe instability unless additional structural conditions are imposed. Durdiev and Rahmonov [4, 5] investigated coefficient identification problems for time-fractional diffusion and wave equations subject to time-nonlocal initial and boundary conditions, together with integral overdetermination conditions providing additional data.

In a recent work, Durdiev [6] considered a multidimensional time-fractional wave equation with integral overdetermination constraints aimed at determining unknown components of a source term assumed to be separable in time and space. Furthermore, Durdiev's monograph [7] provides a comprehensive study of inverse problems related to the identification of source terms and coefficients in fractional-order diffusion equations, and develops a general theoretical framework for their analysis.

Rahmonov [15] investigated the inverse problem of determining a time-dependent coefficient in a time-fractional wave equation posed on a bounded domain, with Robin-type boundary conditions and additional integral (overdetermination) data. In addition, Rahmonov [17] studied an inverse problem for a one-dimensional time-fractional integro-differential wave equation with a Gerasimov–Caputo fractional derivative. The author first analyzed the corresponding direct problem, establishing existence, uniqueness, and regularity of weak solutions under suitable assumptions.

In the theory of inverse problems for fractional integro-differential equations, establishing well-posedness—that is, existence, uniqueness, and continuous dependence of the solution on the data—

is fundamental for both theory and applications. Moreover, Rahmonov [16] examined backward and inverse problems for a time-fractional integro-differential diffusion equation in an abstract Banach space setting, where the temporal dynamics involve nonlocal memory effects through fractional operators.

Backward (or final-value) problems for fractional evolution equations are prototypical examples of ill-posed problems, since reconstructing past states from later measurements generally violates the classical Hadamard well-posedness criteria. The case $1 < \alpha < 2$ differs significantly from the case $0 < \alpha < 1$, which is well posed for any $T > 0$. The main results for backward problems with $\gamma = 0$ (see [9]) can be summarized as follows:

Backward problem in time.

1. $0 < \alpha < 1$: well posed for any $T > 0$;
2. $\alpha = 1$: severely ill posed, but uniqueness and conditional stability hold for any $T > 0$;
3. $1 < \alpha < 2$: well posed for $T > 0$ not belonging to a countable set Λ (see (3.1)); for such exceptional values of T, non-uniqueness may occur.

A recent paper by A.A. Rahmonov (see [18]) addresses an inverse problem for an abstract diffusion equation with a fractional time derivative within an operator framework. A key feature of this work is its abstract setting, which allows the results to apply to a broad class of diffusion-type problems, including those defined on Hilbert and Banach spaces with various boundary conditions.

The use of fractional calculus and spectral methods yields constructive representations of both the direct and inverse solutions, while the application of fixed-point arguments and a priori estimates ensures rigorous well-posedness results, even in the presence of nonlocal effects induced by the fractional time derivative. Rahmonov's contribution fits into the broader framework of fractional inverse problems by extending classical diffusion coefficient recovery to fractional settings, where memory effects complicate both analytical and numerical reconstruction.

The abstract nature of the approach implies that specific partial differential equation models, such as time-fractional diffusion equations on bounded domains, can be treated as particular realizations of the general theory.

The paper is organized as follows. Section 2 introduces auxiliary concepts and new results needed for the study of problem (1.1)–(1.2). In Section 3, we investigate the direct problem in detail, including existence, uniqueness, and regularity of solutions. Section **??** summarizes the main results and presents some open problems related to the problem under consideration. The list of references is given at the end of the paper.

2 Notations and preliminaries

2.1 Functional spaces

In this subsection, we introduce some definitions and preliminary results that will be used throughout the paper.

Let σ be an arbitrary real number. We define the power of the operator A, acting on H, by

$$A^{\sigma} g = \sum_{n=1}^{\infty} \lambda_n^{\sigma} g_n e_n,$$

where $g_n = (g, e_n)$ are the Fourier coefficients of a function $g \in H$. The domain of this operator is given by

$$D(A^\sigma) = \left\{ g \in H : \sum_{n=1}^{\infty} \lambda_n^{2\sigma} |(g, e_n)|^2 < \infty \right\}.$$

For elements of $D(A^\sigma)$, we introduce the norm

$$\|g\|_{D(A^\sigma)}^2 = \sum_{n=1}^{\infty} \lambda_n^{2\sigma} |g_n|^2 = \|A^\sigma g\|^2.$$

Let V be a Banach space. We denote by $C([0,T];V)$ the space of all continuous functions from $[0,T]$ into V, endowed with the norm

$$\|g\|_{C([0,T];V)} = \max_{t\in[0,T]} \|g(t)\|_V.$$

For $0 < \ell < 1$, we denote by $C^{0,\ell}([0,T];V)$ the subspace of $C([0,T];V)$ consisting of all Hölder continuous functions, equipped with the ℓ-Hölder seminorm

$$[g]_{C^{0,\ell}([0,T];V)} := \sup_{0\le t_1<t_2\le T} \frac{\|g(t_2) - g(t_1)\|_V}{|t_2 - t_1|^\ell},$$

and the ℓ-Hölder norm

$$\|g\|_{C^{0,\ell}([0,T];V)} = \|g\|_{C([0,T];V)} + [g]_{C^{0,\ell}([0,T];V)}.$$

We denote by $AC[0,T]$ the space of absolutely continuous functions on $[0,T]$, and define

$$AC^m[0,T] := \{y : y \in C^{m-1}[0,T],\ y^{(m-1)} \in AC[0,T]\}, \quad m \ge 2.$$

The Mittag–Leffler function is defined as follows (see [10], pp. 40–45):

$$E_{\rho,\mu}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\rho k + \mu)}, \quad z \in \mathbb{C},$$

where $\mathrm{Re}(\rho) > 0$ and $\mu \in \mathbb{C}$. It is well known that $E_{\rho,\mu}$ is an entire function on $\mathbb{C}$.

Lemma 2.1 ([14], p. 35)**.** *Let* $0 < \rho < 2$ *and* $\mu \in \mathbb{R}$*. Let* κ *satisfy*

$$\frac{\pi\rho}{2} < \kappa < \min\{\pi, \pi\rho\}.$$

Then there exists a constant $c = c(\rho,\mu,\kappa) > 0$ *such that*

$$\left|E_{\rho,\mu}(z)\right| \le \frac{c}{1 + |z|}, \qquad \kappa \le |\arg(z)| \le \pi.$$

For the proof, we refer to [3].

Lemma 2.2 ([12])**.** *Let* $1 < \alpha < 2$ *and let* $h \in AC[0,T]$*. Define*

$$p(t) = \left(h \star (t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha))\right)(t), \qquad t > 0.$$

If $\lambda > 0$*, then* $p \in AC^2[0,T]$ *and satisfies*

$$\partial_t^\alpha p(t) + \lambda p(t) = h(t), \qquad 0 < t \le T.$$

Lemma 2.3. *Let $1 < \alpha < 2$, $\lambda > 0$, and $m \in \mathbb{N}$. Then the following identities hold:*

For all $t \geq 0$,

$$\frac{d}{dt} E_{\alpha,1}(-\lambda t^\alpha) = -\lambda t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha),$$
$$\frac{d}{dt}\big(t E_{\alpha,2}(-\lambda t^\alpha)\big) = E_{\alpha,1}(-\lambda t^\alpha).$$

For all $t > 0$,

$$\frac{d^m}{dt^m} E_{\alpha,1}(-\lambda t^\alpha) = -\lambda t^{\alpha-m} E_{\alpha,\alpha-m+1}(-\lambda t^\alpha),$$
$$\frac{d}{dt}\big(t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha)\big) = t^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda t^\alpha),$$

and

$$\partial_t^\alpha \big(t E_{\alpha,2}(-\lambda t^\alpha)\big) = -\lambda t E_{\alpha,2}(-\lambda t^\alpha),$$
$$\partial_t^\alpha E_{\alpha,1}(-\lambda t^\alpha) = -\lambda E_{\alpha,1}(-\lambda t^\alpha).$$

The proof can be found in [10].

Lemma 2.4. *Let $1 < \alpha < 2$, $h > 0$, and $t \geq \delta > 0$. Then, for h sufficiently small, the following inequality holds:*

$$\ln\left(1 + \frac{h}{t}\right) \leq \left(\frac{h}{\delta}\right)^{\alpha-1}.$$

The result follows from elementary estimates.

2.2 Solution of the nonlocal problem for an ordinary fractional equation

We consider the fractional ordinary differential equation

$$\partial_t^\alpha y(t) + \lambda y(t) = f(t), \quad 0 < t < T, \tag{2.1}$$

subject to the nonlocal conditions

$$\gamma y(0) + y(T) = a_T, \quad y'(T) = b_T, \tag{2.2}$$

where λ, γ, a_T, and b_T are given real numbers. We assume that $\lambda > 0$ and $\gamma \in (-\infty, -1) \cup (0, \infty)$.

Let f be a given real-valued function defined on $[0, T]$ such that $f \in AC[0, T]$. According to [10, Theorem 5.15, p. 323], the general solution of (2.1) is given by

$$y(t) = E_{\alpha,1}(-\lambda t^\alpha)\, d_1 + t E_{\alpha,2}(-\lambda t^\alpha)\, d_2 + f \star \big(t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha)\big), \tag{2.3}$$

where d_1, d_2 are arbitrary real constants. By Lemma 2.3, for $\lambda > 0$ and $t > 0$, it follows from (2.3) that

$$y'(t) = -\lambda t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha)\, d_1 + E_{\alpha,1}(-\lambda t^\alpha)\, d_2 + f \star \big(t^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda t^\alpha)\big). \tag{2.4}$$

Evaluating (2.3) and (2.4) at $t = T$, we obtain

$$y(T) = E_{\alpha,1}(-\lambda T^\alpha)\, d_1 + T E_{\alpha,2}(-\lambda T^\alpha)\, d_2 + \big(f \star t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha)\big)(T), \tag{2.5}$$

and

$$y'(T) = -\lambda T^{\alpha-1} E_{\alpha,\alpha}(-\lambda T^\alpha)\, d_1 + E_{\alpha,1}(-\lambda T^\alpha)\, d_2 + \big(f \star t^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda t^\alpha)\big)(T). \tag{2.6}$$

Substituting (2.5) and (2.6) into the boundary conditions (2.2), we obtain

$$\begin{aligned} d_1 = \frac{1}{\rho(\lambda T^\alpha)} \Big[& E_{\alpha,1}(-\lambda T^\alpha)\, a_T - T E_{\alpha,2}(-\lambda T^\alpha)\, b_T \\ & - E_{\alpha,1}(-\lambda T^\alpha)\, (f \star t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha))(T) \\ & + T E_{\alpha,2}(-\lambda T^\alpha)\, (f \star t^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda t^\alpha))(T) \Big], \end{aligned}$$

and

$$\begin{aligned} d_2 = \frac{1}{\rho(\lambda T^\alpha)} \Big[& (\gamma + E_{\alpha,1}(-\lambda T^\alpha))\, b_T + \lambda T^{\alpha-1} E_{\alpha,\alpha}(-\lambda T^\alpha)\, a_T \\ & - (\gamma + E_{\alpha,1}(-\lambda T^\alpha))\, (f \star t^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda t^\alpha))(T) \\ & - \lambda T^{\alpha-1} E_{\alpha,\alpha}(-\lambda T^\alpha)\, (f \star t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha))(T) \Big], \end{aligned}$$

where

$$\rho(\eta) = \gamma E_{\alpha,1}(-\eta) + E_{\alpha,1}(-\eta)^2 + \eta\, E_{\alpha,2}(-\eta) E_{\alpha,\alpha}(-\eta), \qquad \eta > 0. \tag{2.7}$$

Substituting the constants d_1 and d_2 into (2.3), we obtain

$$\begin{aligned} y(t) = \frac{E_{\alpha,1}(-\lambda t^\alpha)}{\rho(\lambda T^\alpha)} \Big[& E_{\alpha,1}(-\lambda T^\alpha)\, a_T - T E_{\alpha,2}(-\lambda T^\alpha)\, b_T \\ & - E_{\alpha,1}(-\lambda T^\alpha) \big(f \star t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha)\big)(T) \\ & + T E_{\alpha,2}(-\lambda T^\alpha) \big(f \star t^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda t^\alpha)\big)(T) \Big] \\ + \frac{t E_{\alpha,2}(-\lambda t^\alpha)}{\rho(\lambda T^\alpha)} \Big[& (\gamma + E_{\alpha,1}(-\lambda T^\alpha))\, b_T + \lambda T^{\alpha-1} E_{\alpha,\alpha}(-\lambda T^\alpha)\, a_T \\ & - (\gamma + E_{\alpha,1}(-\lambda T^\alpha)) \big(f \star t^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda t^\alpha)\big)(T) \\ & - \lambda T^{\alpha-1} E_{\alpha,\alpha}(-\lambda T^\alpha) \big(f \star t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha)\big)(T) \Big] \\ + f \star \big(& t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha) \big). \end{aligned} \tag{2.8}$$

Before studying problem (1.1)–(1.2), we analyze the zeros of the function $\rho(\eta)$ defined in (2.7).

Lemma 2.5. *Let $\gamma \in (-\infty, -1) \cup (0, \infty)$. Then the set*

$$\mathrm{P} = \{\eta > 0 : \rho(\eta) = 0\}$$

is nonempty and finite. Moreover, there exists a constant $c_1 > 0$ such that

$$|\rho(\eta)| \geq \frac{c_1}{\eta}, \quad \textit{for all } \eta \geq c_1.$$

Proof. We distinguish two cases.

Case 1: $\gamma < -1$. By the analyticity of the Mittag–Leffler function, the function $\rho(\eta)$ is analytic for $\eta > 0$ and continuous on $[0, \infty)$. Moreover, from the asymptotic expansions of Mittag–Leffler functions for $1 < \alpha < 2$, we have

$$\begin{cases} E_{\alpha,1}(-\eta) = \dfrac{1}{\Gamma(1-\alpha)} \dfrac{1}{\eta} + O\left(\dfrac{1}{\eta^2}\right), \\ E_{\alpha,2}(-\eta) = \dfrac{1}{\Gamma(2-\alpha)} \dfrac{1}{\eta} + O\left(\dfrac{1}{\eta^2}\right), \\ E_{\alpha,\alpha}(-\eta) = \dfrac{-1}{\Gamma(-\alpha)} \dfrac{1}{\eta^2} + O\left(\dfrac{1}{\eta^3}\right), \end{cases} \qquad \text{as } \eta \to \infty, \tag{2.9}$$

(see, e.g., [14], pp. 29–37). Consequently,

$$\rho(\eta) = \gamma E_{\alpha,1}(-\eta) + E_{\alpha,1}(-\eta)^2 + \eta E_{\alpha,2}(-\eta)E_{\alpha,\alpha}(-\eta)$$
$$= \frac{\gamma}{\Gamma(1-\alpha)}\frac{1}{\eta} + O\left(\frac{1}{\eta^2}\right), \quad \text{as } \eta \to \infty.$$

Since $\Gamma(1-\alpha) < 0$ and $\gamma < -1$, it follows that

$$\frac{\gamma}{\Gamma(1-\alpha)} > 0.$$

Hence, there exists $M > 0$ such that $\rho(\eta) > 0$ for all $\eta \geq M$. On the other hand, since $\rho(0) = \gamma + 1 < 0$, the continuity of ρ on $[0,\infty)$ ensures the existence of $\varepsilon > 0$ such that $\rho(\eta) < 0$ for $0 \leq \eta \leq \varepsilon$. By the intermediate value theorem, there exists $\eta_1 \in (\varepsilon, M)$ such that $\rho(\eta_1) = 0$.

Moreover, since ρ is analytic on $[\varepsilon, M]$, the set

$$\{\eta \in [\varepsilon, M] : \rho(\eta) = 0\}$$

is finite. Otherwise, by analyticity, ρ would vanish identically on $[\varepsilon, M]$, which would imply $\rho(0) = 0$, contradicting $\rho(0) = \gamma + 1$.

Case 2: $\gamma > 0$. In this case, we have

$$\frac{\gamma}{\Gamma(1-\alpha)} < 0.$$

Arguing as above, we conclude that the set P is nonempty and finite. Indeed, since $\rho(0) = \gamma + 1 > 0$ and $\rho(\eta) < 0$ for sufficiently large η, the intermediate value theorem again guarantees the existence of at least one zero.

Finally, from the asymptotic expansion (2.9), there exists $c_1 > 0$ such that

$$|\rho(\eta)| \geq \frac{c_1}{\eta}, \qquad \text{for all sufficiently large } \eta. \tag{2.10}$$

This completes the proof. □

We set

$$\{\eta_1, \dots, \eta_N\} = \{\eta > 0 : \rho(\eta) = 0\},$$

with $\eta_1 < \cdots < \eta_N$. By Lemma 2.5, we have no explicit information on the number N of zeros of ρ, except that it is finite and nonzero. In the next lemma, we provide an upper bound for the largest zero η_N.

For convenience, we set

$$\mu_1 := \frac{1}{\alpha\Gamma(-\alpha)}, \quad \mu_2 := \frac{1}{\alpha(\alpha-1)\Gamma(-\alpha)}, \quad \mu_3 := \frac{1}{\Gamma(-\alpha)}.$$

Then $\mu_1, \mu_2, \mu_3 > 0$, since $\Gamma(-\alpha) > 0$ for $1 < \alpha < 2$.

For $1 < \alpha < 2$, we fix θ such that

$$\frac{\pi\alpha}{2} < \theta < \pi.$$

Let $l = l(\theta)$ (see Fig. 1) denote the contour in $\mathbb{C}$ directed from $\infty e^{-i\theta}$ to $\infty e^{i\theta}$, consisting of (see [3], p. 126)

$$l(\theta) = \begin{cases} S_1 = \{z : \arg z = -\theta,\ |z| \geq 1\}, \\ C = \{z : -\theta \leq \arg z \leq \theta,\ |z| = 1\}, \\ S_2 = \{z : \arg z = \theta,\ |z| \geq 1\}. \end{cases}$$

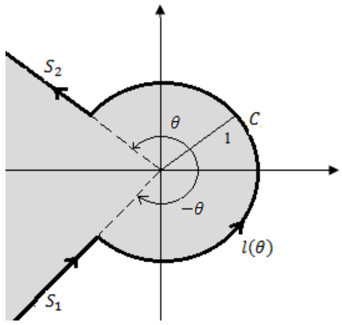

Figure 1: The contour of integration $l(\theta)$.

We define

$$\nu_1 = \frac{1}{2\pi\alpha\sin\theta}\int_l \left|\exp(\zeta^{1/\alpha})\right| |\zeta|\, |d\zeta|,$$

$$\nu_2 = \frac{1}{2\pi\alpha\sin\theta}\int_l \left|\exp(\zeta^{1/\alpha})\right| |\zeta^{1-1/\alpha}|\, |d\zeta|,$$

$$\nu_3 = \frac{1}{2\pi\alpha\sin\theta}\int_l \left|\exp(\zeta^{1/\alpha})\right| |\zeta^{1+1/\alpha}|\, |d\zeta|.$$

Since there exists a constant $c_0 > 0$ such that

$$|\exp(\zeta^{1/\alpha})| \le \exp\left(-c_0|\zeta|^{1/\alpha}\right), \qquad \zeta \in l$$

(see [3], p. 135), it follows that

$$0 < \nu_1, \nu_2, \nu_3 < \infty.$$

We are now in a position to prove the following result.

Lemma 2.6. *Let* $\gamma > 0$. *Then*

$$\eta_N < \max\left\{\frac{1}{|\cos\theta|}, \frac{1}{\mu_1\gamma}\left(\frac{1}{\alpha^2(\alpha-1)\Gamma(-\alpha)} + \gamma\nu_1 + 2\mu_1\nu_1 + \mu_2\nu_3 + \mu_3\nu_2 + \nu_1^2 + \nu_2\nu_3\right)\right\}.$$

Proof. By formula (1.145) in [14], we have

$$\begin{cases} E_{\alpha,1}(-\eta) = -\dfrac{\mu_1}{\eta} + I_{\alpha,1}(\eta), \quad E_{\alpha,2}(-\eta) = \dfrac{\mu_2}{\eta} + I_{\alpha,2}(\eta), \\ E_{\alpha,\alpha}(-\eta) = -\dfrac{\mu_3}{\eta^2} + I_{\alpha,\alpha}(\eta), \quad \eta \ge 1, \end{cases} \tag{2.11}$$

where

$$I_{\alpha,j}(\eta) = \frac{-1}{2\pi\alpha i\,\eta}\int_l \exp\left(\zeta^{\frac{1}{\alpha}}\right)\zeta^{\frac{1-j}{\alpha}+1}\frac{d\zeta}{\zeta+\eta}, \qquad j = 1, 2,$$

and

$$I_{\alpha,\alpha}(\eta) = \frac{1}{2\pi\alpha i\,\eta^2}\int_l \exp\left(\zeta^{\frac{1}{\alpha}}\right)\zeta^{\frac{1}{\alpha}+1}\frac{d\zeta}{\zeta+\eta}, \qquad \eta \ge 1.$$

By [9], we have

$$|I_{\alpha,1}(\eta)| \le \frac{\nu_1}{\eta^2}, \quad |I_{\alpha,2}(\eta)| \le \frac{\nu_2}{\eta^2}, \quad |I_{\alpha,\alpha}(\eta)| \le \frac{\nu_3}{\eta^3}, \qquad \text{for } \eta \ge \frac{1}{|\cos\theta|}. \tag{2.12}$$

Applying (2.11) and (2.12) in (2.7), we obtain, for $\eta \geq \frac{1}{|\cos\theta|} \geq 1$,

$$\begin{aligned}
\rho(\eta) &= \gamma E_{\alpha,1}(-\eta) + E_{\alpha,1}(-\eta)^2 + \eta E_{\alpha,2}(-\eta)E_{\alpha,\alpha}(-\eta) \\
&= -\frac{\mu_1\gamma}{\eta} + \gamma I_{\alpha,1}(\eta) + \left(\frac{\mu_1}{\eta} - I_{\alpha,1}(\eta)\right)^2 \\
&\quad + \left(\frac{\mu_2}{\eta} + I_{\alpha,2}(\eta)\right)\left(-\frac{\mu_3}{\eta} + \eta I_{\alpha,\alpha}(\eta)\right) \\
&= -\frac{\mu_1\gamma}{\eta} + \frac{1}{\eta^2}(\mu_1^2 - \mu_2\mu_3) \\
&\quad + \Big[(\gamma - 2\frac{\mu_1}{\eta})I_{\alpha,1}(\eta) + I_{\alpha,1}(\eta)^2 + \mu_2 I_{\alpha,\alpha}(\eta) \\
&\qquad - \frac{\mu_3 I_{\alpha,2}(\eta)}{\eta} + \eta I_{\alpha,2}(\eta)I_{\alpha,\alpha}(\eta)\Big] \\
&\leq -\frac{\mu_1\gamma}{\eta} + \frac{1}{\eta^2}|\mu_1^2 - \mu_2\mu_3| \\
&\quad + \Big[(\gamma + 2\frac{\mu_1}{\eta})|I_{\alpha,1}(\eta)| + I_{\alpha,1}(\eta)^2 + \mu_2|I_{\alpha,\alpha}(\eta)| \\
&\qquad + \frac{\mu_3|I_{\alpha,2}(\eta)|}{\eta} + \eta|I_{\alpha,2}(\eta)||I_{\alpha,\alpha}(\eta)|\Big] \\
&\leq -\frac{\mu_1\gamma}{\eta} + \frac{1}{\alpha^2(\alpha-1)\Gamma(-\alpha)}\frac{1}{\eta^2} + \frac{\gamma\nu_1}{\eta^2} \\
&\quad + (2\mu_1\nu_1 + \mu_2\nu_3 + \mu_3\nu_2)\frac{1}{\eta^3} + (\nu_1^2 + \nu_2\nu_3)\frac{1}{\eta^4} \\
&\leq -\frac{\mu_1\gamma}{\eta} + \Big[\frac{1}{\alpha^2(\alpha-1)\Gamma(-\alpha)} + \gamma\nu_1 + 2\mu_1\nu_1 \\
&\qquad + \mu_2\nu_3 + \mu_3\nu_2 + \nu_1^2 + \nu_2\nu_3\Big]\frac{1}{\eta^2}.
\end{aligned}$$

Hence,

$$\rho(\eta) = -\frac{\mu_1\gamma}{\eta}\left\{1 - \frac{1}{\mu_1\gamma}\left(\frac{1}{\alpha^2(\alpha-1)\Gamma(-\alpha)} + \gamma\nu_1 + 2\mu_1\nu_1 + \mu_2\nu_3 + \mu_3\nu_2 + \nu_1^2 + \nu_2\nu_3\right)\frac{1}{\eta}\right\}$$

for $\eta \geq \frac{1}{|\cos\theta|}$. This completes the proof of Lemma 2.6. □

Remark 2.7. In the remaining cases for γ, no upper bound for the largest zero η_N is available.

3 Solvability of the FVP (1.1)–(1.2)

This section is devoted to the construction of the solution, the derivation of a priori estimates, and the analysis of uniqueness and regularity properties.

3.1 Construction of the solution and a priori estimates

We introduce the set Λ as follows:

$$\Lambda = \Lambda(\alpha, A) := \bigcup_{n=1}^{\infty}\left\{\left(\frac{\eta_1}{\lambda_n}\right)^{\frac{1}{\alpha}}, \ldots, \left(\frac{\eta_N}{\lambda_n}\right)^{\frac{1}{\alpha}}\right\}. \tag{3.1}$$

Observe that Λ is a countably infinite set. Since $\lambda_n \to \infty$ as $n \to \infty$, the set Λ has 0 as an accumulation point. Moreover, one readily verifies that

$$\Lambda \subset \left[0, \left(\frac{\eta_N}{\lambda_1}\right)^{\frac{1}{\alpha}}\right].$$

Instead of proceeding formally from the abstract equation, we construct the solution as follows. First, for each $n \in \mathbb{N}$, we consider the scalar fractional differential equation for the unknown coefficients $u_n(t)$

$$\partial_t^\alpha u_n(t) + \lambda_n u_n(t) = F_n(t), \qquad t \in (0, T),$$

subject to the nonlocal conditions derived from (1.2),

$$\gamma u_n(0) + u_n(T) = \varphi_n, \quad u_n'(T) = \psi_n, \tag{3.2}$$

where

$$\varphi_n = (\varphi, e_n), \qquad \psi_n = (\psi, e_n), \qquad F_n(t) = (F(t), e_n).$$

Let $T \notin \Lambda$. By Subsection 2.2 and the final-value data (3.2), we obtain

$$\begin{aligned} u_n(t) = {} & \frac{E_{\alpha,1}(-\lambda_n t^\alpha)}{\rho(\lambda_n T^\alpha)} \Big[E_{\alpha,1}(-\lambda_n T^\alpha)\varphi_n - T E_{\alpha,2}(-\lambda_n T^\alpha)\psi_n \\ & \quad - E_{\alpha,1}(-\lambda_n T^\alpha)\big(F_n \star t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^\alpha)\big)(T) \\ & \quad + T E_{\alpha,2}(-\lambda_n T^\alpha)\big(F_n \star t^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda_n t^\alpha)\big)(T) \Big] \\ & + \frac{t E_{\alpha,2}(-\lambda_n t^\alpha)}{\rho(\lambda_n T^\alpha)} \Big[(\gamma + E_{\alpha,1}(-\lambda_n T^\alpha))\psi_n \\ & \quad + \lambda_n T^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n T^\alpha)\varphi_n \\ & \quad - (\gamma + E_{\alpha,1}(-\lambda_n T^\alpha))\big(F_n \star t^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda_n t^\alpha)\big)(T) \\ & \quad - \lambda_n T^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n T^\alpha)\big(F_n \star t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^\alpha)\big)(T) \Big] \\ & + \big(F_n \star t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^\alpha)\big)(t). \end{aligned}$$

We then define the solution $u(t)$ of the original problem (1.1)–(1.2) by the formal series

$$u(t) = \sum_{n=1}^{\infty} u_n(t) e_n,$$

that is,

$$\begin{aligned}
u(t) &= \sum_{n=1}^{\infty}\Big[E_{\alpha,1}(-\lambda_n T^{\alpha})E_{\alpha,1}(-\lambda_n t^{\alpha}) + \lambda_n T^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n T^{\alpha})\, tE_{\alpha,2}(-\lambda_n t^{\alpha})\Big]\frac{\varphi_n}{\rho(\lambda_n T^{\alpha})}e_n \\
&\quad + \sum_{n=1}^{\infty}\Big[-TE_{\alpha,2}(-\lambda_n T^{\alpha})E_{\alpha,1}(-\lambda_n t^{\alpha}) + (\gamma + E_{\alpha,1}(-\lambda_n T^{\alpha}))\, tE_{\alpha,2}(-\lambda_n t^{\alpha})\Big]\frac{\psi_n}{\rho(\lambda_n T^{\alpha})}e_n \\
&\quad - \sum_{n=1}^{\infty}\Big[E_{\alpha,1}(-\lambda_n T^{\alpha})E_{\alpha,1}(-\lambda_n t^{\alpha}) + \lambda_n T^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n T^{\alpha})\, tE_{\alpha,2}(-\lambda_n t^{\alpha})\Big] \\
&\qquad\qquad \times \frac{\Big(F_n(t)\star t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n t^{\alpha})\Big)(T)}{\rho(\lambda_n T^{\alpha})}e_n \\
&\quad + \sum_{n=1}^{\infty}\Big[TE_{\alpha,2}(-\lambda_n T^{\alpha})E_{\alpha,1}(-\lambda_n t^{\alpha}) - (\gamma + E_{\alpha,1}(-\lambda_n T^{\alpha}))\, tE_{\alpha,2}(-\lambda_n t^{\alpha})\Big] \\
&\qquad\qquad \times \frac{\Big(F_n(t)\star t^{\alpha-2}E_{\alpha,\alpha-1}(-\lambda_n t^{\alpha})\Big)(T)}{\rho(\lambda_n T^{\alpha})}e_n \\
&\quad + \sum_{n=1}^{\infty}F_n(t)\star t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n t^{\alpha})\, e_n.
\end{aligned} \tag{3.3}$$

In the following, we rigorously prove that the above series converges in the space

$$C([0,T];D(A))\cap C^1([0,T];H),$$

thereby justifying the term-by-term application of the operator A and the fractional derivative ∂_t^{α}.

For $g\in C([0,T];H)$ and $\varphi,\psi\in H$, we define the operators $\mathcal{P}_j$, $1\le j\le 5$, by

$$(\mathcal{P}_1 g)(t) := \sum_{n=1}^{\infty}\big(g_n\star t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n t^{\alpha})\big)(t)\, e_n,$$

$$\mathcal{P}_2(t)\varphi := \sum_{n=1}^{\infty}\Big[E_{\alpha,1}(-\lambda_n T^{\alpha})E_{\alpha,1}(-\lambda_n t^{\alpha}) + \lambda_n T^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n T^{\alpha})\, tE_{\alpha,2}(-\lambda_n t^{\alpha})\Big]\frac{\varphi_n}{\rho(\lambda_n T^{\alpha})}e_n,$$

$$\mathcal{P}_3(t)\psi := \sum_{n=1}^{\infty}\Big[-TE_{\alpha,2}(-\lambda_n T^{\alpha})E_{\alpha,1}(-\lambda_n t^{\alpha}) + (\gamma + E_{\alpha,1}(-\lambda_n T^{\alpha}))\, tE_{\alpha,2}(-\lambda_n t^{\alpha})\Big]\frac{\psi_n}{\rho(\lambda_n T^{\alpha})}e_n,$$

$$(\mathcal{P}_4 g)(t) := -\mathcal{P}_2(t)\big((\mathcal{P}_1 g)(T)\big),$$

$$(\mathcal{P}_5 g)(t) := -\mathcal{P}_3(t)\big((\mathcal{P}_1 g)(T)\big).$$

Then the solution u can be represented as

$$u(t) = (\mathcal{P}_1F)(t) + \mathcal{P}_2(t)\varphi + \mathcal{P}_3(t)\psi + (\mathcal{P}_4F)(t) + (\mathcal{P}_5F)(t), \tag{3.4}$$

for $t\in(0,T)$.

We define the operator-valued function $Y_A(t)$ by

$$Y_A(t)h = \sum_{n=1}^{\infty}(h,e_n)\, t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n t^{\alpha})e_n,$$

for $h\in H$.

The following result holds.

Lemma 3.1. *Let $\gamma \in (-\infty,-1)\cup(0,\infty)$ and $T \notin \Lambda$. If $\varphi,\psi \in D(A^{1-\frac{1}{\alpha}})$ and $F \in L^\infty(0,T;D(A^{1-\frac{1}{\alpha}}))$, then there exists a constant $C>0$ such that*

$$\|u\|_{C([0,T];H)} \le C\left(\|\varphi\|_{D(A^{1-\frac{1}{\alpha}})} + \|\psi\|_{D(A^{1-\frac{1}{\alpha}})} + \|F\|_{L^\infty(0,T;D(A^{1-\frac{1}{\alpha}}))}\right),$$

where C depends only on γ, α, and T.

Proof. By Lemma 2.1 and the definition of $(\mathcal{P}_1F)(t)$, as well as the generalized Minkowski inequality, we obtain

$$\begin{aligned}\|(\mathcal{P}_1F)(t)\| &\le \int_0^t \|Y_A(t-s)\|\,\|F(s)\|\,ds\\ &\le c\int_0^t (t-s)^{\alpha-1}\|F(s)\|\,ds\\ &\le c\frac{t^\alpha}{\alpha}\|F\|_{L^\infty(0,T;H)}.\end{aligned}$$

Moreover, by Lemma 2.1 and (2.10), the norm $\|\mathcal{P}_2(t)\varphi\|$ can be estimated as follows:

$$\begin{aligned}\|\mathcal{P}_2(t)\varphi\|^2 &\le 2c^4c_1^{-2}\sum_{n=1}^\infty\left|\frac{\lambda_nT^\alpha}{1+\lambda_nT^\alpha}\frac{1}{1+\lambda_nt^\alpha}\varphi_n\right|^2\\ &\quad+2c^4c_1^{-2}\sum_{n=1}^\infty\left|\frac{\lambda_nT^{2\alpha-1}}{1+\lambda_nT^\alpha}\frac{(\lambda_nt^\alpha)^{\frac{1}{\alpha}}}{1+\lambda_nt^\alpha}\lambda_n^{1-\frac{1}{\alpha}}\varphi_n\right|^2,\end{aligned}\tag{3.5}$$

for all $t\in[0,T]$.

Applying

$$\sup_{0<y<\infty}\frac{y^\theta}{1+y} = \theta^\theta(1-\theta)^{1-\theta},\quad \text{for } 0<\theta<1,\tag{3.6}$$

to (3.5), we obtain

$$\|\mathcal{P}_2(t)\varphi\| \le c_2\|\varphi\|_{D(A^{1-\frac{1}{\alpha}})},\quad t\in[0,T],\tag{3.7}$$

where we used the embedding $D(A^{1-\frac{1}{\alpha}})\subset H$ for $1<\alpha<2$. Similarly, we obtain

$$\|\mathcal{P}_3(t)\psi\| \le c_3\|\psi\|_{D(A^{1-\frac{1}{\alpha}})},\quad t\in[0,T].$$

Next, we estimate $\|(\mathcal{P}_4F)(t)\|$ using the same technique as in (3.5). From

$$(\mathcal{P}_4F)(t) = -\mathcal{P}_2(t)(\mathcal{P}_1F)(T),$$

we obtain

$$\begin{aligned}\|(\mathcal{P}_4F)(t)\|^2 &\le 2c^6c_1^2\sum_{n=1}^\infty\left|\frac{\lambda_nT^\alpha}{1+\lambda_nT^\alpha}\frac{1}{1+\lambda_nt^\alpha}\int_0^T F_n(s)\frac{(T-s)^{\alpha-1}}{1+\lambda_n(T-s)^\alpha}\,ds\right|^2\\ &\quad+2c^6c_1^2T^{2(\alpha-1)}\sum_{n=1}^\infty\left|\frac{\lambda_nT^\alpha}{1+\lambda_nT^\alpha}\frac{(\lambda_nt^\alpha)^{\frac{1}{\alpha}}}{1+\lambda_nt^\alpha}\int_0^T F_n(s)\frac{(\lambda_n(T-s)^\alpha)^{\frac{\alpha-1}{\alpha}}}{1+\lambda_n(T-s)^\alpha}\,ds\right|^2\\ &\le 2c^6c_1^2\left(\int_0^T\left(\sum_{n=1}^\infty|F_n(s)|^2\right)^{\frac{1}{2}}(T-s)^{\alpha-1}\,ds\right)^2\\ &\quad+\frac{2c^6c_1^2}{\alpha^4}(\alpha-1)^{\frac{4\alpha-4}{\alpha}}T^{2(\alpha-1)}\left(\int_0^T\left(\sum_{n=1}^\infty|F_n(s)|^2\right)^{\frac{1}{2}}ds\right)^2\\ &\le c_4^2\|F\|^2_{L^\infty(0,T;H)}.\end{aligned}$$

A similar computation shows that

$$\|(\mathcal{P}_5F)(t)\|^2 \le 2c^6c_1^2\sum_{n=1}^{\infty}\left|\frac{\lambda_n T^{\alpha+1}}{1+\lambda_n T^\alpha}\frac{1}{1+\lambda_n t^\alpha}\int_0^T F_n(s)(T-s)^{\alpha-2}ds\right|^2$$
$$+2c^6c_1^2T^{2\alpha}\sum_{n=1}^{\infty}\left|\left(\gamma+\frac{1}{1+\lambda_n T^\alpha}\right)\frac{(\lambda_n t^\alpha)^{\frac{1}{\alpha}}}{1+\lambda_n t^\alpha}\int_0^T \lambda_n^{1-\frac{1}{\alpha}}F_n(s)(T-s)^{\alpha-2}ds\right|^2,$$

and hence

$$\|(\mathcal{P}_5F)(t)\|^2 \le c_5^2\|F\|^2_{L^\infty(0,T;D(A^{1-\frac{1}{\alpha}}))}.$$

This completes the proof of Lemma 3.1. □

3.2 Main result

Based on Lemma 3.1, we establish the existence, uniqueness, and regularity of solutions in the following theorem.

Theorem 3.2. *Let $\gamma \in (-\infty,-1)\cup(0,\infty)$ and $T \notin \Lambda$.*

(1) Let $\varphi,\psi \in D(A^{2-\frac{1}{\alpha}})$ and $F \in L^\infty(0,T;D(A^{2-\frac{1}{\alpha}}))$. Then the FVP (1.1)–(1.2) admits a unique mild solution

$$u \in C([0,T];D(A))\cap C^1([0,T];H)$$

such that $\partial_t^\alpha u \in C((0,T];H)$. Moreover, there exist constants $C_0, C_1 > 0$ such that

$$\|u\|_{C([0,T];D(A))}+\|u'\|_{C([0,T];H)} \le C_0\Big(\|\varphi\|_{D(A^{2-\frac{1}{\alpha}})}+\|\psi\|_{D(A^{2-\frac{1}{\alpha}})}+\|F\|_{L^\infty(0,T;D(A^{2-\frac{1}{\alpha}}))}\Big) \quad (3.8)$$

and, for all $t>0$,

$$\|\partial_t^\alpha u(t)\| \le C_1\Big[\big(t^{-\alpha}+t^{1-\alpha}\big)\big(\|\varphi\|_{D(A)}+\|\psi\|_{D(A)}\big)+\big(1+t^\alpha+t^{1-\alpha}\big)\|F\|_{L^\infty(0,T;D(A))}\Big].$$

(2) Let $\varphi \in D(A)$, $\psi \in D(A^{2-\frac{1}{\alpha}})$, and $F \in AC([0,T];D(A^{2-\frac{1}{\alpha}}))$. Then the FVP (1.1)–(1.2) admits a unique mild solution

$$u \in AC^2([0,T];H)\cap C([0,T];D(A))$$

such that $\partial_t^\alpha u \in L^\infty(0,T;H)$, and

$$\|\partial_t^\alpha u\|_{L^\infty(0,T;H)} \le C_2\Big(\|\varphi\|_{D(A)}+\|\psi\|_{D(A^{2-\frac{1}{\alpha}})}+\|F\|_{L^\infty(0,T;D(A^{2-\frac{1}{\alpha}}))}\Big), \quad (3.9)$$

where $C_2>0$ depends only on α, γ, and T.

(3) Let $\varphi,\psi \in D(A)$ and $F \in C^{0,\alpha-1}([0,T];D(A))$. Then, for the mild solution u given by (3.4), we have, for every $\delta>0$,

$$\|Au\|_{C^{0,\alpha-1}([0,T];H)}+\|\partial_t^\alpha u\|_{C^{0,\alpha-1}([0,T];H)}$$
$$\le C_3\Big[\|F\|_{C^{0,\alpha-1}([0,T];D(A))}+\Big(\frac{1}{\delta^{\alpha-1}}+\frac{1}{\delta^{2\alpha-2}}\Big)\big(\|\varphi\|_{D(A)}+\|\psi\|_{D(A)}+\|F\|_{C([0,T];D(A))}\big)\Big],$$

where $C_3>0$ is independent of φ, ψ, and F.

Proof. We divide the proof of Theorem 3.2 into several steps.

In the first step, we assume that problem (1.1), (1.2) has a solution and show that this solution is unique. The uniqueness of u depends on the uniqueness of the problem (2.1)–(2.2). Indeed, suppose that we have two solutions $u_1(t)$ and $u_2(t)$, and set $v(t) = u_1(t) - u_2(t)$. Then we have

$$\begin{cases} \partial_t^\alpha v(t) + Av(t) = 0, & 0 < t < T, \\ \gamma v(0) + v(T) = 0, \quad v'(T) = 0. \end{cases} \tag{3.10}$$

Set $v_n(t) = (v(t), e_n)$. It follows from (3.10) that, for any $n \in \mathbb{N}$,

$$\partial_t^\alpha v_n(t) = (\partial_t^\alpha v(t), e_n) = -(Av(t), e_n) = -(v(t), Ae_n) = -\lambda_n v_n(t),$$

and the final-time conditions imply

$$\gamma v_n(0) + v_n(T) = 0, \quad v_n'(T) = 0.$$

Therefore, we obtain the following FVP for $v_n(t)$:

$$\begin{cases} \partial_t^\alpha v_n(t) + \lambda_n v_n(t) = 0, & 0 < t < T, \\ \gamma v_n(0) + v_n(T) = 0, \quad v_n'(T) = 0. \end{cases} \tag{3.11}$$

By our assumption, $T \notin \Lambda$. Then, by (2.8), problem (3.11) has only the trivial solution. Therefore, by Parseval's identity, we obtain $v(t) \equiv 0$ on $[0, T]$. Hence, uniqueness is proved.

Furthermore, if

$$\varphi \in D(A^{2-\frac{1}{\alpha}}), \qquad \psi \in D(A^{2-\frac{1}{\alpha}}), \qquad F \in L^\infty(0, T; D(A^{2-\frac{1}{\alpha}})),$$

then $u \in C([0, T]; D(A))$, and the following limits hold:

$$\lim_{t\to 0} \|\gamma u(t) + u(T-t) - \varphi\|_{D(A)} = 0, \qquad \lim_{t\to 0} \|u'(T-t) - \psi\| = 0. \tag{3.12}$$

Indeed, by Lemma 2.1, we have

$$\begin{aligned} \|(\mathcal{P}_1 F)(t)\|_{D(A)} &= \|A(\mathcal{P}_1 F)(t)\| \\ &\overset{\text{g.M.ineq.}}{\leq} \int_0^t \|Y_A(t-s)AF(s)\| ds \\ &\leq c \int_0^t (t-s)^{\alpha-1} \|F(s)\|_{D(A)} ds \\ &\leq c \frac{t^\alpha}{\alpha} \|F\|_{L^\infty(0,T;D(A))}. \end{aligned} \tag{3.13}$$

Similarly, from

$$\|\mathcal{P}_2(t)\varphi\|_{D(A)} = \|A\mathcal{P}_2(t)\varphi\|,$$

and arguing as in the proof of (3.7), we obtain

$$\begin{aligned} \|\mathcal{P}_2(t)\varphi\|_{D(A)}^2 \leq 2c^4 c_1^{-2} \sum_{n=1}^\infty \left| \frac{\lambda_n T^\alpha}{1 + \lambda_n T^\alpha} \frac{\lambda_n}{1 + \lambda_n t^\alpha} \varphi_n \right|^2 & \\ + 2c^4 c_1^{-2} \sum_{n=1}^\infty \left| \frac{\lambda_n T^{2\alpha-1}}{1 + \lambda_n T^\alpha} \frac{\lambda_n t}{1 + \lambda_n t^\alpha} \lambda_n \varphi_n \right|^2 & \\ \leq 2c^4 c_1^{-2} \sum_{n=1}^\infty |\lambda_n \varphi_n|^2 + 2c^4 c_1^{-2} T^{2(\alpha-1)} \sum_{n=1}^\infty \left| \frac{(\lambda_n t^\alpha)^{\frac{1}{\alpha}}}{1 + \lambda_n t^\alpha} \lambda_n^{2-\frac{1}{\alpha}} \varphi_n \right|^2 & \\ \leq c_6^2 \|\varphi\|_{D(A^{2-\frac{1}{\alpha}})}^2, \quad t \geq 0. & \end{aligned} \tag{3.14}$$

For $\|\mathcal{P}_3(t)\psi\|_{D(A)}$, we have

$$\begin{aligned}\|\mathcal{P}_3(t)\psi\|_{D(A)}^2 &\le 2c^4c_1^{-2}\sum_{n=1}^{\infty}\left|\frac{\lambda_n T^{\alpha+1}}{1+\lambda_n T^{\alpha}}\frac{\lambda_n}{1+\lambda_n t^{\alpha}}\psi_n\right|^2\\ &\quad + 2c^4c_1^{-2}T^{2\alpha}\sum_{n=1}^{\infty}\left|\left(\gamma+\frac{1}{1+\lambda_n T^{\alpha}}\right)\frac{(\lambda_n t^{\alpha})^{\frac{1}{\alpha}}}{1+\lambda_n t^{\alpha}}\lambda_n^{2-\frac{1}{\alpha}}\psi_n\right|^2\\ &\le c_7^2\|\psi\|^2_{D(A^{2-\frac{1}{\alpha}})}, \quad t\ge 0.\end{aligned} \tag{3.15}$$

Next, we estimate $\|(\mathcal{P}_4F)(t)\|_{D(A)}$ using the same arguments as in (3.14). By Lemma 2.1 and (3.6), we obtain

$$\begin{aligned}\|(\mathcal{P}_4F)(t)\|_{D(A)}^2 &\le 2c^6c_1^{-2}\sum_{n=1}^{\infty}\left|\frac{\lambda_n T^{\alpha}}{1+\lambda_n T^{\alpha}}\frac{1}{1+\lambda_n t^{\alpha}}\int_0^T\lambda_n^{\frac{1}{\alpha}}F_n(s)\frac{(\lambda_n(T-s)^{\alpha})^{\frac{\alpha-1}{\alpha}}}{1+\lambda_n(T-s)^{\alpha}}ds\right|^2\\ &\quad + 2c^6c_1^{-2}T^{2(\alpha-1)}\sum_{n=1}^{\infty}\left|\frac{\lambda_n T^{\alpha}}{1+\lambda_n T^{\alpha}}\frac{(\lambda_n t^{\alpha})^{\frac{1}{\alpha}}}{1+\lambda_n t^{\alpha}}\int_0^T\lambda_n F_n(s)\frac{(\lambda_n(T-s)^{\alpha})^{\frac{\alpha-1}{\alpha}}}{1+\lambda_n(T-s)^{\alpha}}ds\right|^2\\ &\overset{\text{g.M.ineq.}}{\le} 2c^6c_1^{-2}\left|\int_0^T\left(\sum_{n=1}^{\infty}(\lambda_n^{\frac{1}{\alpha}}F_n(s))^2\right)^{\frac{1}{2}}ds\right|^2\\ &\quad + \frac{2c^6c_1^{-2}}{\alpha^4}(\alpha-1)^{\frac{4\alpha-4}{\alpha}}T^{2(\alpha-1)}\left|\int_0^T\left(\sum_{n=1}^{\infty}(\lambda_n F_n(s))^2\right)^{\frac{1}{2}}ds\right|^2\\ &\le c_7^2\|F\|^2_{L^{\infty}(0,T;D(A^{1/\alpha}))}+c_8^2\|F\|^2_{L^{\infty}(0,T;D(A))}, \quad t\ge 0.\end{aligned} \tag{3.16}$$

Similarly, one can show that

$$\|(\mathcal{P}_5F)(t)\|_{D(A)} \le c_9\left[\|F\|_{L^{\infty}(0,T;D(A))}+\|F\|_{L^{\infty}(0,T;D(A^{2-\frac{1}{\alpha}}))}\right], \qquad t\ge 0. \tag{3.17}$$

Combining (3.13)–(3.17), we obtain

$$\|u(t)\|_{D(A)} \le c_{10}\Big(\|\varphi\|_{D(A^{2-\frac{1}{\alpha}})}+\|\psi\|_{D(A^{2-\frac{1}{\alpha}})}+\|F\|_{L^{\infty}(0,T;D(A^{2-\frac{1}{\alpha}}))}\Big), \qquad t\ge 0. \tag{3.18}$$

Now, in order to prove the first limit in (3.12), we rewrite the expression $\gamma u(t)+u(T-t)-\varphi$ as

$$\begin{aligned}\gamma u(t)+u(T-t)-\varphi &= \gamma\Big[(\mathcal{P}_1F)(t)+\mathcal{P}_2(t)\varphi+\mathcal{P}_3(t)\psi+(\mathcal{P}_4F)(t)+(\mathcal{P}_5F)(t)\Big]\\ &\quad + (\mathcal{P}_1F)(T-t)+\mathcal{P}_2(T-t)\varphi+\mathcal{P}_3(T-t)\psi\\ &\quad + (\mathcal{P}_4F)(T-t)+(\mathcal{P}_5F)(T-t)-\sum_{n=1}^{\infty}\varphi_n e_n.\end{aligned} \tag{3.19}$$

Using (3.3) and (3.18), we obtain

$$\begin{aligned}&\|\gamma u(t)+u(T-t)-\varphi\|_{D(A)}\\ &\le c_{10}(|\gamma|+1)\Big(\|\varphi\|_{D(A^{2-\frac{1}{\alpha}})}+\|\psi\|_{D(A^{2-\frac{1}{\alpha}})}+\|F\|_{L^{\infty}(0,T;D(A^{2-\frac{1}{\alpha}}))}\Big)+\|\varphi\|_{D(A)}\end{aligned}$$

for all $t\ge 0$.

Thus, the functional series in (3.19) is uniformly bounded in $D(A)$ for all $t \geq 0$. Hence, we may pass to the limit termwise as $t \to 0$, which yields the first limit in (3.12).

Now, we show that $u \in C^1([0,T];H)$. Indeed, by Lemma 2.3, we have

$$\partial_t(\mathcal{P}_1 F)(t) = \sum_{n=1}^{\infty} F_n(t) \star t^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda_n t^\alpha) e_n,$$

and a combination of the generalized Minkowski inequality and Parseval's identity yields

$$\begin{aligned}
\|\partial_t(\mathcal{P}_1 F)(t)\| &\leq \int_0^t \left\| \sum_{n=1}^{\infty} (t-s)^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda_n (t-s)^\alpha) F_n(s) e_n \right\| ds \\
&\leq \int_0^t \left[\sum_{n=1}^{\infty} \left(F_n(s) \frac{(t-s)^{\alpha-2}}{1+\lambda_n (t-s)^\alpha} \right)^2 \right]^{1/2} ds \\
&\leq \frac{t^{\alpha-1}}{\alpha-1} \|F\|_{L^\infty(0,T;H)}.
\end{aligned} \tag{3.20}$$

Applying Lemma 2.3 once more, we obtain

$$\begin{aligned}
\partial_t(\mathcal{P}_2(t)\varphi) = \sum_{n=1}^{\infty} \Big[&-\lambda_n t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^\alpha) E_{\alpha,1}(-\lambda_n T^\alpha) \\
&+ \lambda_n T^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n T^\alpha) E_{\alpha,1}(-\lambda_n t^\alpha) \Big] \frac{\varphi_n}{\rho(\lambda_n T^\alpha)} e_n.
\end{aligned}$$

Then, by virtue of (2.10) and (3.6), we obtain

$$\begin{aligned}
\|\partial_t(\mathcal{P}_2(t)\varphi)\|^2 &= \sum_{n=1}^{\infty} \Big| \Big[-\lambda_n t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^\alpha) E_{\alpha,1}(-\lambda_n T^\alpha) \\
&\qquad + \lambda_n T^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n T^\alpha) E_{\alpha,1}(-\lambda_n t^\alpha) \Big] \frac{\varphi_n}{\rho(\lambda_n T^\alpha)} \Big|^2 \\
&\leq 2c^4 c_1^{-2} \sum_{n=1}^{\infty} \frac{\lambda_n^2 T^{2\alpha}}{(1+\lambda_n T^\alpha)^2} \left(\frac{(\lambda_n t^\alpha)^{\frac{\alpha-1}{\alpha}}}{1+\lambda_n t^\alpha} \right)^2 \lambda_n^{\frac{2}{\alpha}} \varphi_n^2 \\
&\quad + 2c^4 c_1^{-2} \sum_{n=1}^{\infty} \frac{\lambda_n^2}{(1+\lambda_n t^\alpha)^2} \left(\frac{\lambda_n^2 T^{2\alpha-1}}{1+\lambda_n T^\alpha} \right)^2 \varphi_n^2.
\end{aligned}$$

By (2.12) and the embedding $D(A) \subset D(A^{\frac{1}{\alpha}})$ for $1 < \alpha < 2$, we obtain

$$\|\partial_t(\mathcal{P}_2(t)\varphi)\| \leq c_{11} \|\varphi\|_{D(A)}. \tag{3.21}$$

Similarly, we have

$$\|\partial_t(\mathcal{P}_3(t)\psi)\| \leq c_{12} \|\psi\|_{D(A)}. \tag{3.22}$$

Furthermore, by defining $\mathcal{P}_4 F(t)$ and using Lemma 2.3, we have

$$\begin{aligned}
\partial_t(\mathcal{P}_4 F)(t) &= -\partial_t \big(\mathcal{P}_2(t)(\mathcal{P}_1 F)(T)\big) \\
&= \sum_{n=1}^{\infty} \frac{\lambda_n}{\rho(\lambda_n T^\alpha)} \Big[E_{\alpha,1}(-\lambda_n T^\alpha)\, t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^\alpha) - T^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n T^\alpha) E_{\alpha,1}(-\lambda_n t^\alpha) \Big] \\
&\quad \times \Big(F_n(t) \star t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^\alpha) \Big)(T)\, e_n \\
&:= \mathcal{I}_{11} F(t) + \mathcal{I}_{12} F(t).
\end{aligned}$$

Then, similarly to (3.15), we get

$$
\begin{aligned}
&\|\mathcal{I}_{11}F(t)\|^2\\
&=\sum_{n=1}^{\infty}\left|\frac{\lambda_n}{\rho(\lambda_n T^\alpha)}E_{\alpha,1}(-\lambda_n T^\alpha)t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n t^\alpha)\int_0^T F_n(s)(T-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(T-s)^\alpha)\,ds\right|^2\\
&\le c^6c_1^{-2}\sum_{n=1}^{\infty}\frac{\lambda_n^2T^{2\alpha}}{(1+\lambda_nT^\alpha)^2}\left(\frac{(\lambda_nt^\alpha)^{\frac{\alpha-1}{\alpha}}}{1+\lambda_nt^\alpha}\right)^2\left|\int_0^T F_n(s)\frac{(\lambda_n(T-s)^\alpha)^{\frac{1}{\alpha}}}{1+\lambda_n(T-s)^\alpha}(T-s)^{\alpha-2}\,ds\right|^2\\
&\le c^6c_1^{-2}(\alpha-1)^{\frac{4\alpha-4}{\alpha}}\left(\frac{1}{\alpha}\right)^4\left|\int_0^T\left(\sum_{n=1}^{\infty}\left(F(s,\cdot),e_n\right)^2\right)^{1/2}(T-s)^{\alpha-2}\,ds\right|^2\\
&\le c_{13}^2\|F\|^2_{L^\infty(0,T;H)}.
\end{aligned}
\tag{3.23}
$$

Similarly, we have

$$
\|\mathcal{I}_{12}F(t,\cdot)\|\le c_{14}\|F\|_{L^\infty(0,T;D(A^{\frac{1}{\alpha}}))}. \tag{3.24}
$$

Once more, using Lemma 2.3, we get

$$
\begin{aligned}
\partial_t(\mathcal{P}_5F)(t)&=-\partial_t(\mathcal{P}_3(t)F(T))\\
&=\sum_{n=1}^{\infty}\frac{\lambda_n}{\rho(\lambda_nT^\alpha)}\Big[TE_{\alpha,2}(-\lambda_nT^\alpha)\,t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_nt^\alpha)+(\gamma+E_{\alpha,1}(-\lambda_nT^\alpha))E_{\alpha,1}(-\lambda_nt^\alpha)\Big]\\
&\quad\times\left(F_n(r)\star r^{\alpha-2}E_{\alpha,\alpha-1}(-\lambda_nr^\alpha)\right)\Big|_{r=T}e_n\\
&:=\mathcal{I}_{13}F(t)+\mathcal{I}_{14}F(t).
\end{aligned}
$$

Then, similarly to (3.17), we obtain

$$
\begin{aligned}
\|\mathcal{I}_{13}F(t)\|^2&=\sum_{n=1}^{\infty}\Bigg|\frac{\lambda_nT}{\rho(\lambda_nT^\alpha)}E_{\alpha,2}(-\lambda_nT^\alpha)t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_nt^\alpha)\\
&\qquad\times\int_0^T F_n(s)(T-s)^{\alpha-2}E_{\alpha,\alpha-1}(-\lambda_n(T-s)^\alpha)ds\Bigg|^2\\
&\le c^6c_1^{-2}\sum_{n=1}^{\infty}\frac{\lambda_n^4T^{2\alpha+2}}{(1+\lambda_nT^\alpha)^2}\frac{t^{2(\alpha-1)}}{(1+\lambda_nt^\alpha)^2}\left|\int_0^T F_n(s)\frac{(T-s)^{\alpha-2}}{1+\lambda_n(T-s)^\alpha}ds\right|^2\\
&\le c_0^6c_1^2T^2(\alpha-1)^{\frac{2\alpha-2}{\alpha}}\left(\frac{1}{\alpha}\right)^2\left|\int_0^T\left(\sum_{n=1}^{\infty}(\lambda_n^{\frac{1}{\alpha}}F_n(s))^2\right)^{1/2}(T-s)^{\alpha-2}ds\right|^2\\
&\le c_{15}^2\|F\|^2_{L^\infty(0,T;D(A^{\frac{1}{\alpha}}))}.
\end{aligned}
\tag{3.25}
$$

Arguing as in (3.25), we obtain for $\mathcal{I}_{14}F(t)$ that

$$
\|\mathcal{I}_{14}F(t)\|\le c_{16}\|F\|_{L^\infty(0,T;D(A))}. \tag{3.26}
$$

It follows from (3.13)–(3.17) and (3.20)–(3.26) that $u\in C^1([0,T];H)$. Moreover, the second limit in (3.12) follows analogously. Consequently, inequality (3.8) is a direct consequence of (3.18) and (3.20)–(3.26).

Furthermore, assuming that $\varphi, \psi \in D(A)$ and $F \in L^\infty(0,T;D(A))$, we obtain, for all $t > 0$,

$$\|Au(t)\| \le c'_{10}\left(t^{-\alpha} + t^{1-\alpha}\right)\left(\|\varphi\|_{D(A)} + \|\psi\|_{D(A)}\right) + c'_{11}\left(1 + t^{\alpha} + t^{1-\alpha}\right)\|F\|_{L^\infty(0,T;D(A))}. \tag{3.27}$$

In addition, using (1.1) together with (3.27), we obtain, for all $t > 0$,

$$\|\partial_t^\alpha u(t)\| \le c_{17}\left(t^{-\alpha} + t^{1-\alpha}\right)\left(\|\varphi\|_{D(A)} + \|\psi\|_{D(A)}\right) + c_{18}\left(1 + t^{\alpha} + t^{1-\alpha}\right)\|F\|_{L^\infty(0,T;D(A))}.$$

In the following, we improve the regularity of the solution u so that $\partial_t^\alpha u$ is well defined in the sense of the Caputo fractional derivative.

By Lemma 2.2, we have $(\mathcal{P}_1 F)(t) \in AC([0,T];H)$ and

$$\begin{aligned}\partial_t^\alpha(\mathcal{P}_1 F)(t) &= \sum_{n=1}^\infty \left(F_n(t) - \lambda_n \int_0^t F_n(s)(t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-s)^\alpha)\,ds\right)e_n \\ &= F(t) - A(\mathcal{P}_1 F)(t).\end{aligned} \tag{3.28}$$

It follows from (3.28) and (3.13) that $\partial_t^\alpha(\mathcal{P}_1 F) \in C([0,T];H)$.

Next, by Lemma 2.3, the second derivative of $\mathcal{P}_2(t)\varphi$ is given by

$$\begin{aligned}\partial_{tt}(\mathcal{P}_2(t)\varphi) = \sum_{n=1}^\infty \Big[&-\lambda_n t^{\alpha-2}E_{\alpha,\alpha-1}(-\lambda_n t^\alpha)E_{\alpha,1}(-\lambda_n T^\alpha) \\ &- \lambda_n^2 T^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n T^\alpha)t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n t^\alpha)\Big]\frac{\varphi_n}{\rho(\lambda_n T^\alpha)}e_n.\end{aligned}$$

Using (2.10), (2.12), and Parseval's identity, we obtain

$$\begin{aligned}\|\partial_{tt}(\mathcal{P}_2(t)\varphi)\|^2 &\le 2c^4c_1^{-2}\sum_{n=1}^\infty\left|\frac{t^{\alpha-2}}{1+\lambda_n t^\alpha}\frac{1}{1+\lambda_n T^\alpha}\lambda_n^2\varphi_n\right|^2 \\ &\quad + 2c^4c_1^{-2}\sum_{n=1}^\infty\left|\frac{t^{\alpha-1}}{1+\lambda_n t^\alpha}\frac{\lambda_n T^{\alpha-1}}{1+\lambda_n T^\alpha}\lambda_n^2\varphi_n\right|^2 \\ &\le c_{19}^2 t^{2(\alpha-2)}\|\varphi\|_{D(A)}^2 + c_{20}^2 t^{2(\alpha-2)}\|\varphi\|_{D(A)}^2.\end{aligned} \tag{3.29}$$

Since $1 < \alpha < 2$, it follows that

$$\partial_{tt}(\mathcal{P}_2(t)\varphi) \in L^1(0,T;H), \quad \mathcal{P}_2(t)\varphi \in AC^2([0,T];H).$$

From Lemma 2.3, we know that the second derivative of $\mathcal{P}_3(t)\psi$ satisfies

$$\begin{aligned}\partial_{tt}(\mathcal{P}_3(t)\psi) = \sum_{n=1}^\infty \lambda_n\Big[&TE_{\alpha,2}(-\lambda_n T^\alpha)\,t^{\alpha-2}E_{\alpha,\alpha-1}(-\lambda_n t^\alpha) \\ &- \left(\gamma + E_{\alpha,1}(-\lambda_n T^\alpha)\right)t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n t^\alpha)\Big]\frac{\psi_n}{\rho(\lambda_n T^\alpha)}e_n.\end{aligned}$$

Then, by Lemma 2.1 and (3.6), we obtain

$$\|\partial_{tt}(\mathcal{P}_3(t)\psi)\| \le c_{21}t^{\alpha-2}\|\psi\|_{D(A)} + c_{19}t^{\alpha-2}\|\psi\|_{D(A^{2-\frac{1}{\alpha}})}. \tag{3.30}$$

Since $1 < \alpha < 2$, it follows that

$$\partial_{tt}(\mathcal{P}_3(t)\psi) \in L^1(0,T;H), \quad \mathcal{P}_3(t)\psi \in AC^2([0,T];H).$$

Furthermore, for the second derivative of $(\mathcal{P}_4 F)(t)$, we obtain

$$\begin{aligned}\partial_{tt}(\mathcal{P}_4 F)(t) = \sum_{n=1}^{\infty} \frac{\lambda_n}{\rho(\lambda_n T^\alpha)} \Big[& E_{\alpha,1}(-\lambda_n T^\alpha) t^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda_n t^\alpha) \\ & + \lambda_n T^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n T^\alpha) t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^\alpha) \Big] \\ & \times \Big(F_n(t) \star t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^\alpha) \Big)(T) e_n \\ := \mathcal{I}_{15} F(t) + \mathcal{I}_{16} F(t). \end{aligned}$$

Then, similarly to (3.15), we obtain

$$\begin{aligned}\|\mathcal{I}_{15} F(t)\|^2 &= \sum_{n=1}^{\infty} \Bigg| \frac{\lambda_n}{\rho(\lambda_n T^\alpha)} E_{\alpha,1}(-\lambda_n T^\alpha) t^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda_n t^\alpha) \\ &\qquad \times \int_0^T F_n(s)(T-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (T-s)^\alpha)\, ds \Bigg|^2 \\ &\le c^6 c_1^{-2} \sum_{n=1}^{\infty} \frac{\lambda_n^2 T^{2\alpha}}{(1+\lambda_n T^\alpha)^2} \left(\frac{t^{\alpha-2}}{1+\lambda_n t^\alpha} \right)^2 \left| \int_0^T \lambda_n F_n(s) \frac{(T-s)^{\alpha-1}}{1+\lambda_n (T-s)^\alpha}\, ds \right|^2 \\ &\le c^6 c_1^{-2} t^{2(\alpha-2)} \left| \int_0^T \left(\sum_{n=1}^{\infty} (AF(s), e_n)^2 \right)^{1/2} (T-s)^{\alpha-1}\, ds \right|^2 \\ &\le c_{22}^2\, t^{2(\alpha-2)} \|F\|^2_{L^\infty(0,T;D(A))}. \end{aligned} \tag{3.31}$$

Similarly, we obtain

$$\|\mathcal{I}_{16} F(t)\| \le c_{23}\, t^{\alpha-2}\, \|F\|_{L^\infty(0,T;D(A))}. \tag{3.32}$$

Since $1 < \alpha < 2$, it follows that

$$\partial_{tt}\big(\mathcal{P}_4 F(t)\big) \in L^1(0,T;H), \quad \mathcal{P}_4 F(t) \in AC^2([0,T];H).$$

Further, by Lemma 2.3, we obtain

$$\begin{aligned}\partial_{tt}(\mathcal{P}_5 F)(t) = -\sum_{n=1}^{\infty} \frac{\lambda_n}{\rho(\lambda_n T^\alpha)} \Big[& T E_{\alpha,2}(-\lambda_n T^\alpha) t^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda_n t^\alpha) \\ & - \big(\gamma + E_{\alpha,1}(-\lambda_n T^\alpha)\big) t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^\alpha) \Big] \\ & \times \Big(F_n(r) \star r^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda_n r^\alpha) \Big)(T) e_n \\ := \mathcal{I}_{17} F(t) + \mathcal{I}_{18} F(t). \end{aligned}$$

Then

$$
\begin{aligned}
\|\mathcal{I}_{17}F(t)\|^2 &= \sum_{n=1}^{\infty}\left|\frac{\lambda_n}{\rho(\lambda_n T^\alpha)} T E_{\alpha,2}(-\lambda_n T^\alpha) t^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda_n t^\alpha)\right.\\
&\qquad \left.\times \int_0^T F_n(s)(T-s)^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda_n (T-s)^\alpha)\,ds\right|^2\\
&\le c^6 c_1^{-2} \sum_{n=1}^{\infty} \frac{\lambda_n^2 T^{2\alpha+2}}{(1+\lambda_n T^\alpha)^2}\left(\frac{t^{\alpha-2}}{1+\lambda_n t^\alpha}\right)^2 \left|\int_0^T \lambda_n F_n(s)\frac{(T-s)^{\alpha-2}}{1+\lambda_n (T-s)^\alpha}\,ds\right|^2\\
&\le c^6 c_1^{-2} t^{2(\alpha-2)} \left|\int_0^T \left(\sum_{n=1}^{\infty}(AF(s),e_n)^2\right)^{1/2} (T-s)^{\alpha-2}\,ds\right|^2\\
&\le c_{24}^2\, t^{2(\alpha-2)} \|F\|^2_{L^\infty(0,T;D(A))}.
\end{aligned}
\tag{3.33}
$$

Similarly to (3.33), we obtain

$$
\|\mathcal{I}_{18}F(t)\| \le c_{24}\, t^{\alpha-2}\, \|F\|_{L^\infty(0,T;D(A^{2-\frac{1}{\alpha}}))}. \tag{3.34}
$$

Since $1<\alpha<2$, it follows that

$$
\partial_{tt}\big(\mathcal{P}_5 F(t)\big) \in L^1(0,T;H), \quad \mathcal{P}_5 F(t) \in AC^2([0,T];H).
$$

Combining (3.28)–(3.34) together with (1.1), we obtain

$$
\partial_t^\alpha u(t) = -Au(t) + F(t) \in L^\infty(0,T;H).
$$

Therefore, $u \in AC^2([0,T];H)$, and estimate (3.9) follows directly.

We now prove that $u \in C^{0,\alpha-1}([0,T];H)$. Let $0 \le t < t+h \le T$. Then

$$
\begin{aligned}
u(t+h)-u(t) &= \big[(\mathcal{P}_1F)(t+h) - (\mathcal{P}_1F)(t)\big] + \big[\mathcal{P}_2(t+h)\varphi - \mathcal{P}_2(t)\varphi\big]\\
&\quad + \big[\mathcal{P}_3(t+h)\psi - \mathcal{P}_3(t)\psi\big] + \big[(\mathcal{P}_4F)(t+h) - (\mathcal{P}_4F)(t)\big]\\
&\quad + \big[(\mathcal{P}_5F)(t+h) - (\mathcal{P}_5F)(t)\big]\\
&=: \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3 + \mathrm{I}_4 + \mathrm{I}_5.
\end{aligned}
$$

Using the differentiation identities

$$
\partial_t E_{\alpha,1}(-\lambda_n t^\alpha) = -\lambda_n t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^\alpha), \quad \partial_t\big(t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n t^\alpha)\big) = t^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda_n t^\alpha),
$$

we obtain

$$
\begin{aligned}
\mathrm{I}_1 &= \sum_{n=1}^{\infty}\left[\int_t^{t+h} F_n(s)(t+h-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (t+h-s)^\alpha)\,ds\right] e_n\\
&\quad + \sum_{n=1}^{\infty}\left[\int_0^t F_n(s)\Big((t+h-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t+h-s)^\alpha)\right.\\
&\qquad\qquad \left. - (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-s)^\alpha)\Big)ds\right] e_n\\
&= \sum_{n=1}^{\infty}\left[\int_t^{t+h} F_n(s)(t+h-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (t+h-s)^\alpha)\,ds\right] e_n\\
&\quad + \sum_{n=1}^{\infty}\left[\int_0^t \int_{t-s}^{t+h-s} F_n(s)\tau^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda_n \tau^\alpha)\,d\tau\,ds\right] e_n\\
&:= \mathrm{I}_{11} + \mathrm{I}_{12}.
\end{aligned}
\tag{3.35}
$$

We now establish estimates for I_j, $j = 1, 2$, and show that $\mathrm{I}_j \to 0$ as $h \to 0$.

By Lemma 2.1 and (3.6), we obtain

$$\begin{aligned}
\|\mathrm{I}_{11}(t)\|^2_{D(A)} &\le c^2 \sum_{n=1}^{\infty} \left| \int_t^{t+h} \lambda_n^{1-\frac{1}{\alpha}} F_n(s) \frac{(\lambda_n (t+h-s)^\alpha)^{\frac{1}{\alpha}}}{1+\lambda_n (t+h-s)^\alpha} (t+h-s)^{\alpha-2}\, ds \right|^2 \\
&\le c^2 \frac{(\alpha-1)^{\frac{2\alpha-2}{\alpha}}}{\alpha^2} \left(\int_t^{t+h} \left(\sum_{n=1}^{\infty} \left(\lambda_n^{1-\frac{1}{\alpha}} F_n(s)\right)^2 \right)^{1/2} (t+h-s)^{\alpha-2}\, ds \right)^2 \\
&\le c^2 \frac{(\alpha-1)^{\frac{2\alpha-2}{\alpha}}}{\alpha^2} \|F\|^2_{C([0,T];D(A^{1-\frac{1}{\alpha}}))}\, h^{2(\alpha-1)}.
\end{aligned}$$

This yields

$$\|\mathrm{I}_{11}(t)\|_{D(A)} \le c_{25} \|F\|_{C([0,T];D(A^{1-\frac{1}{\alpha}}))}\, h^{\alpha-1},$$

where $c_{25} = \dfrac{(\alpha-1)^{\frac{\alpha-1}{\alpha}}}{\alpha} c$.

Next, we estimate the term I_{12}. Using Lemma 2.1, we obtain

$$\|\mathrm{I}_{12}(t)\|^2_{D(A)} \le c^2 \sum_{n=1}^{\infty} \left| \int_0^t \lambda_n F_n(s) \left(\int_{t-s}^{t+h-s} \tau^{\alpha-2}\, d\tau \right) ds \right|^2.$$

For $0 < s < t$, we have

$$\int_{t-s}^{t+h-s} \tau^{\alpha-2}\, d\tau = \frac{1}{\alpha-1} \left[(t+h-s)^{\alpha-1} - (t-s)^{\alpha-1} \right].$$

We note that the estimate

$$(t+h-s)^{\alpha-1} - (t-s)^{\alpha-1} \le h^{\alpha-1}$$

follows from the concavity of the function $f(t) = t^\kappa$ with $0 < \kappa < 1$, namely,

$$f(t_1) - f(t_2) \le f(t_1 - t_2), \quad 0 < t_2 < t_1.$$

Therefore, we deduce

$$\|\mathrm{I}_{12}(t)\|^2_{D(A)} \le \frac{c^2}{(\alpha-1)^2} \sum_{n=1}^{\infty} \left| \int_0^t \lambda_n F_n(s)\, ds \right|^2 h^{2\alpha-2},$$

which implies

$$\|\mathrm{I}_{12}(t)\|_{D(A)} \le c_{26} \|F\|_{C([0,T];D(A))}\, h^{\alpha-1}.$$

Furthermore, we have

$$\begin{aligned}
\mathrm{I}_2 &= \sum_{n=1}^{\infty} \left[E_{\alpha,1}(-\lambda_n T^\alpha) \big(E_{\alpha,1}(-\lambda_n (t+h)^\alpha) - E_{\alpha,1}(-\lambda_n t^\alpha) \big) \right] \frac{\varphi_n}{\rho(\lambda_n T^\alpha)} e_n \\
&\quad + \sum_{n=1}^{\infty} \left[\lambda_n T^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n T^\alpha) \big((t+h) E_{\alpha,2}(-\lambda_n (t+h)^\alpha) - t E_{\alpha,2}(-\lambda_n t^\alpha) \big) \right] \frac{\varphi_n}{\rho(\lambda_n T^\alpha)} e_n \\
&=: \mathrm{I}_{21} + \mathrm{I}_{22}.
\end{aligned}$$

To estimate I_{21}, we use Lemmas 2.1 and 2.3. We obtain

$$|E_{\alpha,1}(-\lambda_n(t+h)^\alpha) - E_{\alpha,1}(-\lambda_n t^\alpha)| = \left|\int_t^{t+h} \lambda_n \tau^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n \tau^\alpha)\, d\tau\right| \leq \int_t^{t+h} \lambda_n \tau^{\alpha-1} \frac{c}{1+\lambda_n \tau^\alpha}\, d\tau \leq c \int_t^{t+h} \tau^{-1}\, d\tau. \quad (3.36)$$

Then, for $\delta \leq t \leq T$, we have

$$\begin{aligned} \|\mathrm{I}_{21}(t)\|^2_{D(A)} &\leq c^2 c_1^{-2} \sum_{n=1}^{\infty} \left| \frac{\lambda_n^2 T^\alpha}{1+\lambda_n T^\alpha} \varphi_n \int_t^{t+h} \tau^{-1}\, d\tau \right|^2 \\ &\leq c^2 c_1^{-2} \left(\ln\left(1+\frac{h}{t}\right)\right)^2 \sum_{n=1}^{\infty} \lambda_n^2 \varphi_n^2 \\ &\leq c^2 c_1^{-2} \frac{h^2}{\delta^2} \|\varphi\|^2_{D(A)} \\ &\leq c_{27}^2 \left(\frac{h}{\delta}\right)^{2(\alpha-1)} \|\varphi\|^2_{D(A)}. \end{aligned} \quad (3.37)$$

Here we have used Lemma 2.4.

We now estimate I_{22}. Using Lemmas 2.1 and 2.3, we obtain

$$\begin{aligned} \|\mathrm{I}_{22}(t)\|^2_{D(A)} &\leq c^2 c_1^{-2} \sum_{n=1}^{\infty} \left| \frac{\lambda_n^2 T^{2\alpha-1}}{1+\lambda_n T^\alpha} \varphi_n \int_t^{t+h} E_{\alpha,1}(-\lambda_n s^\alpha)\, ds \right|^2 \\ &\leq c^4 c_1^{-2} T^{2(\alpha-1)} \sum_{n=1}^{\infty} \varphi_n^2 \left| \int_t^{t+h} s^{-\alpha}\, ds \right|^2 \\ &\leq \frac{c^4 c_1^{-2}}{(\alpha-1)^2} T^{2(\alpha-1)} \left| \frac{(t+h)^{\alpha-1} - t^{\alpha-1}}{(t+h)^{\alpha-1} t^{\alpha-1}} \right|^2 \|\varphi\|^2. \end{aligned}$$

We note that

$$(t+h)^{\alpha-1} - t^{\alpha-1} \leq h^{\alpha-1},$$

and, for $\delta \leq t \leq T$,

$$(t+h)^{\alpha-1} t^{\alpha-1} \geq \delta^{2\alpha-2}.$$

Hence, we obtain

$$\|\mathrm{I}_{22}(t)\|_{D(A)} \leq c_{28} \frac{h^{\alpha-1}}{\delta^{2\alpha-2}} \|\varphi\|. \quad (3.38)$$

Similarly to (3.36)–(3.38), we obtain the following estimate for I_3:

$$\|\mathrm{I}_3(t,\cdot)\|_{D(A)} \leq c_{29} \frac{h^{\alpha-1}}{\delta^{\alpha-1}} \|\psi\| + c_{30} \frac{h^{\alpha-1}}{\delta^{2\alpha-2}} \|\psi\|_{D(A)}. \quad (3.39)$$

We now estimate I_4. We write

$$\begin{aligned} \mathrm{I}_4 = &-\sum_{n=1}^{\infty} \Big[E_{\alpha,1}(-\lambda_n T^\alpha)\big(E_{\alpha,1}(-\lambda_n(t+h)^\alpha) - E_{\alpha,1}(-\lambda_n t^\alpha)\big) \\ &+ \lambda_n T^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n T^\alpha)\big((t+h)E_{\alpha,2}(-\lambda_n(t+h)^\alpha) - t E_{\alpha,2}(-\lambda_n t^\alpha)\big)\Big] \\ &\times \frac{\big(F_n(t) \star t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^\alpha)\big)(T)}{\rho(\lambda_n T^\alpha)} e_n \\ := &\ \mathrm{I}_{41} + \mathrm{I}_{42}. \end{aligned}$$

For I_{41}, we have

$$\begin{aligned}
\|\mathrm{I}_{41}(t)\|^2_{D(A)} &\le c_{27}^2 \frac{h^{2(\alpha-1)}}{\delta^{2(\alpha-1)}} \sum_{n=1}^{\infty} \left| \frac{\lambda_n T^\alpha}{1+\lambda_n T^\alpha} \int_0^T \lambda_n^{1-\frac{1}{\alpha}} F_n(s) \frac{(\lambda_n (T-s)^\alpha)^{\frac{1}{\alpha}}}{1+\lambda_n (T-s)^\alpha} (T-s)^{\alpha-2}\, ds \right|^2 \\
&\le c_{27}^2 \frac{(\alpha-1)^{\frac{2\alpha-2}{\alpha}}}{\alpha^2} \frac{h^{2(\alpha-1)}}{\delta^{2(\alpha-1)}} \left| \int_0^T \left(\sum_{n=1}^{\infty} (\lambda_n^{1-\frac{1}{\alpha}} F_n(s))^2 \right)^{1/2} (T-s)^{\alpha-2}\, ds \right|^2 \\
&\le c_{31}^2 \frac{h^{2(\alpha-1)}}{\delta^{2(\alpha-1)}} \|F\|^2_{C([0,T];D(A^{1-\frac{1}{\alpha}}))}.
\end{aligned}$$

Similarly, we obtain

$$\|\mathrm{I}_{42}(t)\|_{D(A)} \le c_{32} \frac{h^{\alpha-1}}{\delta^{2\alpha-2}} \|F\|_{C([0,T];D(A))}.$$

Indeed, applying Lemmas 2.1 and 2.3, together with the generalized Minkowski integral inequality and (3.38), we obtain

$$\begin{aligned}
\|\mathrm{I}_{22}(t)\|^2_{D(A)} &= \sum_{n=1}^{\infty} \lambda_n^2 \left| \frac{\lambda_n T^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n T^\alpha)}{\rho(\lambda_n T^\alpha)} \int_t^{t+h} E_{\alpha,1}(-\lambda_n s^\alpha)\, ds \right. \\
&\qquad \left. \times \int_0^T F_n(\eta)(T-\eta)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (T-\eta)^\alpha)\, d\eta \right|^2 \\
&\le c^6 c_1^{-2} \sum_{n=1}^{\infty} \left| \frac{\lambda_n^2 T^{\alpha-1}}{1+\lambda_n T^\alpha} \cdot \lambda_n T^\alpha \int_t^{t+h} \frac{1}{1+\lambda_n s^\alpha}\, ds \right. \\
&\qquad \left. \times \int_0^T F_n(\eta) \frac{(T-\eta)^{\alpha-1}}{1+\lambda_n (T-\eta)^\alpha}\, d\eta \right|^2 \\
&\le c^6 c_1^{-2} \sum_{n=1}^{\infty} \left| \lambda_n T^{\alpha-1} \int_t^{t+h} s^{-\alpha}\, ds \int_0^T F_n(\eta)(T-\eta)^{\alpha-1}\, d\eta \right|^2 \\
&\le c^6 c_1^{-2} T^{2(\alpha-1)} \left(\int_t^{t+h} s^{-\alpha}\, ds \right)^2 \left(\int_0^T \left(\sum_{n=1}^{\infty} (\lambda_n F_n(\eta))^2 \right)^{1/2} (T-\eta)^{\alpha-1}\, d\eta \right)^2 \\
&\le \frac{c^6 c_1^{-2}}{(\alpha(\alpha-1))^2} T^{2(2\alpha-1)} \frac{h^{2(\alpha-1)}}{\delta^{4(\alpha-1)}} \|F\|^2_{C([0,T];D(A))},
\end{aligned}$$

and we can prove

$$\|\mathrm{I}_5(t)\|_{D(A)} \le c_{33} \frac{h^{\alpha-1}}{\delta^{\alpha-1}} \|F\|_{C([0,T];D(A))} + c_{34} \frac{h^{\alpha-1}}{\delta^{2\alpha-2}} \|F\|_{C([0,T];D(A))}.$$

Now we complete the proof of Theorem 3.2 (3). By Lemmas 2.2 and 2.3, we have

$$\begin{aligned}
\partial_t^\alpha u(t) &= \sum_{n=1}^{\infty} F_n(t)e_n - \sum_{n=1}^{\infty} \lambda_n F_n(t) \star t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n t^\alpha)e_n \\
&- \sum_{n=1}^{\infty} \lambda_n \Big[E_{\alpha,1}(-\lambda_n T^\alpha)E_{\alpha,1}(-\lambda_n t^\alpha) + \lambda_n T^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n T^\alpha)tE_{\alpha,2}(-\lambda_n t^\alpha)\Big]\frac{\varphi_n}{\rho(\lambda_n T^\alpha)}e_n \\
&- \sum_{n=1}^{\infty} \lambda_n \Big[-TE_{\alpha,2}(-\lambda_n T^\alpha)E_{\alpha,1}(-\lambda_n t^\alpha) + (\gamma + E_{\alpha,1}(-\lambda_n T^\alpha))tE_{\alpha,2}(-\lambda_n t^\alpha)\Big]\frac{\psi_n}{\rho(\lambda_n T^\alpha)}e_n \\
&+ \sum_{n=1}^{\infty} \lambda_n \Big[E_{\alpha,1}(-\lambda_n T^\alpha)E_{\alpha,1}(-\lambda_n t^\alpha) + \lambda_n T^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n T^\alpha)tE_{\alpha,2}(-\lambda_n t^\alpha)\Big] \\
&\quad \times \frac{\big(F_n(t) \star t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n t^\alpha)\big)(T)}{\rho(\lambda_n T^\alpha)}e_n \\
&- \sum_{n=1}^{\infty} \lambda_n \Big[TE_{\alpha,2}(-\lambda_n T^\alpha)E_{\alpha,1}(-\lambda_n t^\alpha) - (\gamma + E_{\alpha,1}(-\lambda_n T^\alpha))tE_{\alpha,2}(-\lambda_n t^\alpha)\Big] \\
&\quad \times \frac{\big(F_n(t) \star t^{\alpha-2}E_{\alpha,\alpha-1}(-\lambda_n t^\alpha)\big)(T)}{\rho(\lambda_n T^\alpha)}e_n.
\end{aligned}$$

Then

$$\begin{aligned}
\partial_t^\alpha u(t+h) - \partial_t^\alpha u(t) &= F(t+h) - F(t) - \Big[(\mathcal{P}_1 AF)(t+h) - (\mathcal{P}_1 AF)(t)\Big] \\
&\quad - \Big[(\mathcal{P}_2(t+h)A\varphi) - (\mathcal{P}_2(t)A\varphi)\Big] - \Big[(\mathcal{P}_3(t+h)A\psi) - (\mathcal{P}_3(t)A\psi)\Big] \\
&\quad + \Big[(\mathcal{P}_4 AF)(t+h) - (\mathcal{P}_4 AF)(t)\Big] - \Big[(\mathcal{P}_5 AF)(t+h) - (\mathcal{P}_5 AF)(t)\Big] \\
&:= \mathrm{I}_6 - \mathrm{I}_7 - \mathrm{I}_8 - \mathrm{I}_9 + \mathrm{I}_{10} - \mathrm{I}_{11}.
\end{aligned} \tag{3.40}$$

Now, we establish estimates for I_j $(j = 6, \ldots, 11)$ and show that I_j tends to 0 as $h \to 0$. The following estimate is obtained easily:

$$\|\mathrm{I}_6(t)\| = \frac{\|F(t+h) - F(t)\|}{h^{\alpha-1}}\, h^{\alpha-1} \le [F]_{C^{0,\alpha-1}([0,T];H)}\, h^{\alpha-1}.$$

By (3.35), the second term in (3.40) can be written as follows:

$$\begin{aligned}
\mathrm{I}_7 &= \sum_{n=1}^{\infty} \lambda_n \left[\int_t^{t+h} F_n(s)(t+h-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t+h-s)^\alpha)\,ds\right]e_n \\
&\quad + \sum_{n=1}^{\infty} \lambda_n \left[\int_0^t \int_{t-s}^{t+h-s} F_n(s)\tau^{\alpha-2}E_{\alpha,\alpha-1}(-\lambda_n \tau^\alpha)\,d\tau\,ds\right]e_n.
\end{aligned}$$

We observe that the last equality differs from (3.35) only by the presence of the factor λ_n. Therefore, we obtain

$$\|\mathrm{I}_7(t)\| \le c_{35}\left(\|F\|_{C([0,T];D(A^{1-\frac{1}{\alpha}}))} + \|F\|_{C([0,T];D(A))}\right)h^{\alpha-1}.$$

Similarly to (3.37) and (3.38), we have

$$\|\mathrm{I}_8(t)\| \le c_{36}\left(\frac{1}{\delta^{\alpha-1}}\|\varphi\|_{D(A)} + \frac{1}{\delta^{2\alpha-2}}\|\varphi\|\right)h^{\alpha-1}.$$

Further, by (3.39), we obtain

$$\|\mathrm{I}_9(t)\| \le c_{37}\left(\frac{1}{\delta^{\alpha-1}}\|\psi\|_{D(A)} + \frac{1}{\delta^{2\alpha-2}}\|\psi\|\right) h^{\alpha-1}.$$

Similarly, we obtain

$$\|\mathrm{I}_{10}(t)\| \le c_{38}\left(\frac{1}{\delta^{\alpha-1}}\|F\|_{C([0,T];D(A^{1-\frac{1}{\alpha}}))} + \frac{1}{\delta^{2\alpha-2}}\|F\|_{C([0,T];H)}\right) h^{\alpha-1}.$$

and

$$\|\mathrm{I}_{11}(t)\| \le c_{39}\left(\frac{1}{\delta^{\alpha-1}}\|F\|_{C([0,T];D(A))} + \frac{1}{\delta^{2\alpha-2}}\|F\|_{C([0,T];D(A))}\right) h^{\alpha-1}.$$

Thus, the proof of Theorem 3.2 is complete. □

Corollary 3.3. *Let $\gamma \in (-\infty,-1) \cup (0,\infty)$ and $T \notin \Lambda$. Let $\varphi, \psi \in D(A)$ and $F = 0$. Then, there exists a constant $C_4 > 0$ such that $u \in C^\infty((0,T];H)$ and*

$$\|\partial_t^m u(t)\| \le \frac{C_4}{t^m}\Big(\|\varphi\| + \|\psi\| + t\big(\|\varphi\|_{D(A)} + \|\psi\|_{D(A)}\big)\Big), \qquad t > 0,\ m \in \mathbb{N}.$$

The proof is a direct consequence of (3.4) and Lemma 2.3.

4 Conclusion

This paper investigates a nonlocal initial-boundary value problem for an abstract time-fractional wave equation involving an unbounded, positive, self-adjoint operator on a Hilbert space. The existence of a mild solution is established via the eigenfunction decomposition method. By expanding the solution in terms of the operator's eigenfunctions, the problem is reduced to a system of ordinary fractional differential equations with nonlocal conditions, whose solutions are expressed using the Mittag–Leffler function. An analysis of the zeros of the denominator determines the admissible interval of definition, excluding the right endpoint. The mild solution is represented as a series expansion in the eigenfunctions of the abstract operator, and estimation techniques in the associated Hilbert spaces are employed to prove existence, uniqueness, and regularity.

An open problem remains regarding the behavior of solutions for certain values of the parameter γ, namely $\gamma \in [-1,0)$. As noted in Remark 2.7, in these cases, there is currently no available information on an upper bound for the largest zero η_N. This lack of control presents significant analytical difficulties in deriving uniform estimates or obtaining further spectral bounds, and it therefore represents a promising direction for future research.

In addition, we briefly mention the open inverse source problems for (1.1)–(1.2) under different overdetermination conditions, which merit further investigation.

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Conflict of Interest

The authors declare that there are no conflicts of interest.

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