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Some New Opial-Like Inequalities for Two Functions and Applications

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Abstract. In this paper, we establish, among other results, that if $f, g : [a, b] \to \mathbb{C}$ are absolutely continuous functions satisfying $f(a) = 0$ and $g(b) = 0$, then

$$\int_a^b |f(t)g(t)|\, dt \leq \frac{1}{4}(b-a)^2 \int_a^b \left(|f'(t)|^2 + |g'(t)|^2\right) dt - \frac{1}{4}\int_a^b \left((t-a)^2 |f'(t)|^2 + (b-t)^2 |g'(t)|^2\right) dt,$$

provided that the integrals on the right-hand side are finite. In particular, if $f(a) = f(b) = 0$, then the following sharp inequality holds:

$$\int_a^b |f(t)|^2\, dt \leq \int_a^b (b-t)(t-a)\, |f'(t)|^2\, dt.$$

We also derive several trapezoid-type and Grüss-type inequalities.

Keywords: Opial's inequality; trapezoid inequality; Grüss inequality
MSC (2020): 26D15, 26D10

1 Introduction

We recall the following Opial-type inequalities.

Theorem 1.1. *Let $u\colon [a, b] \subset \mathbb{R} \to \mathbb{R}$ be an absolutely continuous function on the interval $[a, b]$ such that $u' \in L_2[a, b]$.*

(i) *If $u(a) = u(b) = 0$, then*

$$\int_a^b |u(t)\, u'(t)|\, dt \leq \frac{1}{4}(b-a) \int_a^b |u'(t)|^2\, dt, \tag{1.1}$$

with equality if and only if

$$u(t) = \begin{cases} c(t-a), & a \leq t \leq \dfrac{a+b}{2}, \\ c(b-t), & \dfrac{a+b}{2} < t \leq b, \end{cases}$$

where c is an arbitrary constant.

(ii) *If $u(a) = 0$, then*

$$\int_a^b |u(t)\,u'(t)|\,dt \le \frac{1}{2}(b-a)\int_a^b |u'(t)|^2\,dt, \tag{1.2}$$

with equality if and only if $u(t) = c(t-a)$ for some constant c.

Inequality (1.1) was proved by Olech [7], who provided a simplified proof of an inequality originally due, in a slightly less general form, to Zdzisław Opial [8].

Olech's argument also contains the half-interval version of Opial's inequality, which was independently obtained by Beesack [1] and applies to functions u that vanish only at a. For further proofs and related developments, see [3–6] and [9].

In the recent paper [2], we established the following two-function versions of the Opial-type inequalities stated above.

Theorem 1.2. *Let $f, g\colon [a,b] \to \mathbb{C}$ be absolutely continuous on $[a,b]$ and assume that $f', g' \in L_2[a,b]$.*

(i) *If $g(a) = 0$, then*

$$\begin{aligned}\int_a^b \left|f'(t)\,g(t)\right| dt &\le \left(\int_a^b (t-a)\left|f'(t)\right|^2 dt\right)^{1/2}\left(\int_a^b (b-t)\left|g'(t)\right|^2 dt\right)^{1/2}\\ &\le \frac{1}{2}\left[\int_a^b (t-a)\left|f'(t)\right|^2 dt + \int_a^b (b-t)\left|g'(t)\right|^2 dt\right].\end{aligned}$$

(ii) *If $g(b) = 0$, then*

$$\begin{aligned}\int_a^b \left|f'(t)\,g(t)\right| dt &\le \left(\int_a^b (b-t)\left|f'(t)\right|^2 dt\right)^{1/2}\left(\int_a^b (t-a)\left|g'(t)\right|^2 dt\right)^{1/2}\\ &\le \frac{1}{2}\left[\int_a^b (b-t)\left|f'(t)\right|^2 dt + \int_a^b (t-a)\left|g'(t)\right|^2 dt\right].\end{aligned}$$

(iii) *If $g(a) = g(b) = 0$, then*

$$\begin{aligned}\int_a^b \left|f'(t)\,g(t)\right| dt &\le \left(\int_a^b K(t)\left|f'(t)\right|^2 dt\right)^{1/2}\left(\int_a^b \left|\frac{a+b}{2} - t\right|\left|g'(t)\right|^2 dt\right)^{1/2}\\ &\le \frac{1}{2}\left[\int_a^b K(t)\left|f'(t)\right|^2 dt + \int_a^b \left|\frac{a+b}{2} - t\right|\left|g'(t)\right|^2 dt\right],\end{aligned}$$

where

$$K(t) = \begin{cases} t-a, & a \le t \le \dfrac{a+b}{2},\\ b-t, & \dfrac{a+b}{2} < t \le b.\end{cases}$$

We have the following refinement of the Opial inequalities (1.1) and (1.2).

Corollary 1.3. *Let $f\colon [a,b] \to \mathbb{C}$ be absolutely continuous on $[a,b]$ and assume that $f' \in L_2[a,b]$.*

(i) *If either $f(a) = 0$ or $f(b) = 0$, then*

$$\begin{aligned}\int_a^b \left|f'(t)\,f(t)\right| dt &\le \left(\int_a^b (t-a)\left|f'(t)\right|^2 dt\right)^{1/2}\left(\int_a^b (b-t)\left|f'(t)\right|^2 dt\right)^{1/2}\\ &\le \frac{1}{2}(b-a)\int_a^b \left|f'(t)\right|^2 dt.\end{aligned}$$

(ii) *If* $f(a) = f(b) = 0$, *then*

$$\int_a^b \left|f'(t)\, f(t)\right| dt \le \left(\int_a^b K(t) \left|f'(t)\right|^2 dt\right)^{1/2} \left(\int_a^b \left|\frac{a+b}{2} - t\right| \left|f'(t)\right|^2 dt\right)^{1/2}$$
$$\le \frac{1}{4}(b-a) \int_a^b \left|f'(t)\right|^2 dt.$$

2 Main Results

Let $w : [a,b] \to [0,\infty)$ be a measurable weight. We define the weighted space

$$L_{2,w}[a,b] := \left\{ h : [a,b] \to \mathbb{C} \text{ measurable} : \int_a^b w(t)\,|h(t)|^2\,dt < \infty \right\}.$$

We consider the positive weight function

$$w_a(t;a,b) := \frac{1}{2}\left[(b-a)^2 - (t-a)^2\right] = (b-t)\left(\frac{b+t}{2} - a\right), \qquad t \in [a,b].$$

Theorem 2.1. *Let* $f, g\colon [a,b] \to \mathbb{C}$ *be absolutely continuous. Assume that*

$$f(a) = g(a) = 0, \qquad f', g' \in L_{2,w_a}[a,b].$$

Then

$$\int_a^b \left|f(t)\,g(t)\right| dt \le \left(\int_a^b w_a(t;a,b) \left|f'(t)\right|^2 dt\right)^{1/2} \left(\int_a^b w_a(t;a,b) \left|g'(t)\right|^2 dt\right)^{1/2} \tag{2.1}$$
$$\le \frac{1}{2} \int_a^b w_a(t;a,b) \left(\left|f'(t)\right|^2 + \left|g'(t)\right|^2\right) dt.$$

Moreover, both inequalities in (2.1) *are sharp.*

Proof. Since $f(a) = g(a) = 0$ and f, g are absolutely continuous on $[a,b]$, we have

$$f(t) = \int_a^t f'(s)\,ds \qquad \text{and} \qquad g(t) = \int_a^t g'(s)\,ds \quad \text{for all } t \in [a,b].$$

Hence

$$\int_a^b |f(t)g(t)|\,dt = \int_a^b \left|\int_a^t f'(s)\,ds\right| \left|\int_a^t g'(s)\,ds\right| dt$$
$$= \int_a^b (t-a)\,(t-a)^{-1/2} \left|\int_a^t f'(s)\,ds\right| (t-a)^{-1/2} \left|\int_a^t g'(s)\,ds\right| dt. \tag{2.2}$$

Denote the last integral by A.

By the Cauchy–Bunyakovsky–Schwarz inequality, for every $t \in [a,b]$,

$$(t-a)^{-1/2} \left|\int_a^t f'(s)\,ds\right| \le \left(\int_a^t |f'(s)|^2\,ds\right)^{1/2}, \qquad (t-a)^{-1/2} \left|\int_a^t g'(s)\,ds\right| \le \left(\int_a^t |g'(s)|^2\,ds\right)^{1/2}.$$

Therefore,

$$A \le \int_a^b (t-a) \left(\int_a^t |f'(s)|^2\,ds\right)^{1/2} \left(\int_a^t |g'(s)|^2\,ds\right)^{1/2} dt.$$

Applying again the Cauchy–Bunyakovsky–Schwarz inequality, we obtain

$$A \le \left[\int_a^b (t-a)\left(\int_a^t |f'(s)|^2\,ds\right)dt\right]^{1/2}\left[\int_a^b (t-a)\left(\int_a^t |g'(s)|^2\,ds\right)dt\right]^{1/2}. \tag{2.3}$$

Denote the right-hand side of (2.3) by B.

Using integration by parts, we compute

$$\begin{aligned}\int_a^b (t-a)\left(\int_a^t |f'(s)|^2\,ds\right)dt &= \int_a^b \left(\int_a^t |f'(s)|^2\,ds\right) d\left(\frac{(t-a)^2}{2}\right)\\ &= \left(\int_a^t |f'(s)|^2\,ds\right)\frac{(t-a)^2}{2}\Bigg|_{t=a}^{t=b} - \int_a^b \frac{(t-a)^2}{2}|f'(t)|^2\,dt\\ &= \frac{(b-a)^2}{2}\int_a^b |f'(s)|^2\,ds - \int_a^b \frac{(t-a)^2}{2}|f'(t)|^2\,dt\\ &= \int_a^b \left[\frac{(b-a)^2}{2} - \frac{(t-a)^2}{2}\right]|f'(t)|^2\,dt\\ &= \int_a^b w_a(t;a,b)\,|f'(t)|^2\,dt,\end{aligned}$$

and similarly,

$$\int_a^b (t-a)\left(\int_a^t |g'(s)|^2\,ds\right)dt = \int_a^b w_a(t;a,b)\,|g'(t)|^2\,dt.$$

Consequently,

$$B \le \left(\int_a^b w_a(t;a,b)\,|f'(t)|^2\,dt\right)^{1/2}\left(\int_a^b w_a(t;a,b)\,|g'(t)|^2\,dt\right)^{1/2}. \tag{2.4}$$

Combining (2.2)–(2.4) yields the first inequality in (2.1). The second inequality in (2.1) follows from the arithmetic mean–geometric mean inequality,

$$\sqrt{\alpha\beta} \le \frac{\alpha+\beta}{2}, \qquad \alpha,\beta \ge 0. \tag{2.5}$$

To show sharpness, take $f(t) = g(t) = t - a$. Then

$$\int_a^b |f(t)g(t)|\,dt = \int_a^b (t-a)^2\,dt = \frac{1}{3}(b-a)^3,$$

and, since $f'(t) = g'(t) = 1$,

$$\begin{aligned}\frac{1}{2}\int_a^b w_a(t;a,b)\left(|f'(t)|^2 + |g'(t)|^2\right)dt &= \int_a^b w_a(t;a,b)\,dt\\ &= \frac{1}{2}\int_a^b \left[(b-a)^2 - (t-a)^2\right]dt\\ &= \frac{1}{2}\left[(b-a)^2(b-a) - \frac{(b-a)^3}{3}\right] = \frac{1}{3}(b-a)^3.\end{aligned}$$

Thus equality holds throughout (2.1), which proves that both inequalities are sharp. □

Remark 2.2. Assume that f' is absolutely continuous on $[a,b]$. If $f(a) = f'(a) = 0$ and $f', f'' \in L_{2,w_a}[a,b]$, then

$$\int_a^b \left|f(t)\, f'(t)\right| dt \le \left(\int_a^b w_a(t;a,b) \left|f'(t)\right|^2 dt\right)^{1/2} \left(\int_a^b w_a(t;a,b) \left|f''(t)\right|^2 dt\right)^{1/2}$$
$$\le \frac{1}{2}\int_a^b w_a(t;a,b) \left(\left|f'(t)\right|^2 + \left|f''(t)\right|^2\right) dt.$$

The inequality follows from (2.1) by taking $g = f'$.

Corollary 2.3. *Let $f\colon [a,b] \to \mathbb{C}$ be absolutely continuous. Assume that*

$$f(a) = 0, \qquad f' \in L_{2,w_a}[a,b].$$

Then

$$\int_a^b \left|f(t)\right|^2 dt \le \int_a^b w_a(t;a,b) \left|f'(t)\right|^2 dt. \tag{2.6}$$

Moreover, the inequality in (2.6) is sharp.

Now consider the dual weight function

$$w_b(t;a,b) := \frac{1}{2}\left[(b-a)^2 - (b-t)^2\right] = (t-a)\left(b - \frac{a+t}{2}\right), \qquad t \in [a,b].$$

Theorem 2.4. *Let $f, g\colon [a,b] \to \mathbb{C}$ be absolutely continuous. Assume that*

$$f(b) = g(b) = 0, \qquad f', g' \in L_{2,w_b}[a,b].$$

Then

$$\int_a^b \left|f(t)\, g(t)\right| dt \le \left(\int_a^b w_b(t;a,b) \left|f'(t)\right|^2 dt\right)^{1/2} \left(\int_a^b w_b(t;a,b) \left|g'(t)\right|^2 dt\right)^{1/2} \tag{2.7}$$
$$\le \frac{1}{2}\int_a^b w_b(t;a,b) \left(\left|f'(t)\right|^2 + \left|g'(t)\right|^2\right) dt.$$

Moreover, the inequalities in (2.7) are sharp.

Proof. Since $f(b) = g(b) = 0$ and f, g are absolutely continuous on $[a,b]$, we have

$$f(t) = -\int_t^b f'(s)\, ds \qquad \text{and} \qquad g(t) = -\int_t^b g'(s)\, ds, \quad t \in [a,b].$$

Hence

$$\int_a^b |f(t)g(t)|\, dt = \int_a^b \left|\int_t^b f'(s)\, ds\right| \left|\int_t^b g'(s)\, ds\right| dt$$
$$= \int_a^b (b-t)\,(b-t)^{-1/2} \left|\int_t^b f'(s)\, ds\right| (b-t)^{-1/2} \left|\int_t^b g'(s)\, ds\right| dt. \tag{2.8}$$

Denote the last integral by C.

By the Cauchy–Bunyakovsky–Schwarz inequality, for every $t \in [a,b]$,

$$(b-t)^{-1/2} \left|\int_t^b f'(s)\, ds\right| \le \left(\int_t^b |f'(s)|^2\, ds\right)^{1/2}, \qquad (b-t)^{-1/2} \left|\int_t^b g'(s)\, ds\right| \le \left(\int_t^b |g'(s)|^2\, ds\right)^{1/2}.$$

Therefore,

$$C \le \int_a^b (b-t)\left(\int_t^b |f'(s)|^2\,ds\right)^{1/2}\left(\int_t^b |g'(s)|^2\,ds\right)^{1/2} dt.$$

Applying the Cauchy–Bunyakovsky–Schwarz inequality again, we obtain

$$C \le \left[\int_a^b (b-t)\left(\int_t^b |f'(s)|^2\,ds\right) dt\right]^{1/2}\left[\int_a^b (b-t)\left(\int_t^b |g'(s)|^2\,ds\right) dt\right]^{1/2}. \tag{2.9}$$

Denote the right-hand side of (2.9) by D.

Using integration by parts, we compute

$$\begin{aligned}\int_a^b (b-t)\left(\int_t^b |f'(s)|^2\,ds\right) dt &= -\int_a^b \left(\int_t^b |f'(s)|^2\,ds\right) d\left(\frac{(b-t)^2}{2}\right)\\ &= -\left(\int_t^b |f'(s)|^2\,ds\right)\frac{(b-t)^2}{2}\bigg|_{t=a}^{t=b} - \int_a^b \frac{(b-t)^2}{2}\,|f'(t)|^2\,dt\\ &= \frac{(b-a)^2}{2}\int_a^b |f'(s)|^2\,ds - \int_a^b \frac{(b-t)^2}{2}\,|f'(t)|^2\,dt\\ &= \int_a^b \left[\frac{(b-a)^2}{2} - \frac{(b-t)^2}{2}\right]|f'(t)|^2\,dt\\ &= \int_a^b w_b(t;a,b)\,|f'(t)|^2\,dt,\end{aligned}$$

and similarly,

$$\int_a^b (b-t)\left(\int_t^b |g'(s)|^2\,ds\right) dt = \int_a^b w_b(t;a,b)\,|g'(t)|^2\,dt.$$

Consequently,

$$D \le \left(\int_a^b w_b(t;a,b)\,|f'(t)|^2\,dt\right)^{1/2}\left(\int_a^b w_b(t;a,b)\,|g'(t)|^2\,dt\right)^{1/2}. \tag{2.10}$$

Combining (2.8)–(2.10) yields the first inequality in (2.7). The second inequality in (2.7) follows from the arithmetic mean–geometric mean inequality (2.5).

To show sharpness, take $f(t) = g(t) = b - t$ for $t \in [a,b]$. Then equality holds throughout (2.7). □

Remark 2.5. Assume that f' is absolutely continuous on $[a,b]$. If $f(b) = f'(b) = 0$ and $f', f'' \in L_{2,w_b}[a,b]$, then

$$\begin{aligned}\int_a^b \left|f(t)\,f'(t)\right| dt &\le \left(\int_a^b w_b(t;a,b)\left|f'(t)\right|^2 dt\right)^{1/2}\left(\int_a^b w_b(t;a,b)\left|f''(t)\right|^2 dt\right)^{1/2}\\ &\le \frac{1}{2}\int_a^b w_b(t;a,b)\left(\left|f'(t)\right|^2 + \left|f''(t)\right|^2\right) dt.\end{aligned}$$

Corollary 2.6. *Let $f\colon [a,b] \to \mathbb{C}$ be absolutely continuous. Assume that*

$$f(b) = 0, \qquad f' \in L_{2,w_b}[a,b].$$

Then

$$\int_a^b \left|f(t)\right|^2 dt \le \int_a^b w_b(t;a,b)\left|f'(t)\right|^2 dt. \tag{2.11}$$

Moreover, the inequality in (2.11) is sharp.

We also have the following result.

Theorem 2.7. *Let $f, g\colon [a,b] \to \mathbb{C}$ be absolutely continuous. Assume that*

$$f(a) = g(b) = 0, \qquad f' \in L_{2,w_a}[a,b],\ g' \in L_{2,w_b}[a,b].$$

Then

$$\int_a^b \left|f(t)\,g(t)\right| dt \le \left(\int_a^b w_a(t;a,b)\left|f'(t)\right|^2 dt\right)^{1/2}\left(\int_a^b w_b(t;a,b)\left|g'(t)\right|^2 dt\right)^{1/2} \tag{2.12}$$
$$\le \frac{1}{4}(b-a)^2\int_a^b \left(\left|f'(t)\right|^2 + \left|g'(t)\right|^2\right) dt$$
$$-\frac{1}{4}\int_a^b \left((t-a)^2\left|f'(t)\right|^2 + (b-t)^2\left|g'(t)\right|^2\right) dt.$$

Moreover, the inequalities in (2.12) are sharp.

Proof. Since $f(a) = 0$ and $g(b) = 0$, and f, g are absolutely continuous on $[a,b]$, we have

$$f(t) = \int_a^t f'(s)\,ds \qquad \text{and} \qquad g(t) = -\int_t^b g'(s)\,ds, \quad t \in [a,b].$$

Therefore,

$$\int_a^b |f(t)g(t)|\,dt = \int_a^b \left|\int_a^t f'(s)\,ds\right|\left|\int_t^b g'(s)\,ds\right| dt$$
$$= \int_a^b (t-a)^{1/2}(b-t)^{1/2}\,(t-a)^{-1/2}\left|\int_a^t f'(s)\,ds\right|\,(b-t)^{-1/2}\left|\int_t^b g'(s)\,ds\right| dt.$$

Denote the last integral by E.

By the Cauchy–Bunyakovsky–Schwarz inequality, for every $t \in [a,b]$,

$$(t-a)^{-1/2}\left|\int_a^t f'(s)\,ds\right| \le \left(\int_a^t |f'(s)|^2\,ds\right)^{1/2}, \qquad (b-t)^{-1/2}\left|\int_t^b g'(s)\,ds\right| \le \left(\int_t^b |g'(s)|^2\,ds\right)^{1/2}.$$

Hence,

$$E \le \int_a^b (t-a)^{1/2}(b-t)^{1/2}\left(\int_a^t |f'(s)|^2\,ds\right)^{1/2}\left(\int_t^b |g'(s)|^2\,ds\right)^{1/2} dt$$
$$= \int_a^b \left[(t-a)^{1/2}\left(\int_a^t |f'(s)|^2\,ds\right)^{1/2}\right]\left[(b-t)^{1/2}\left(\int_t^b |g'(s)|^2\,ds\right)^{1/2}\right] dt.$$

Applying the Cauchy–Bunyakovsky–Schwarz inequality again, we obtain

$$E \le \left(\int_a^b (t-a)\int_a^t |f'(s)|^2\,ds\,dt\right)^{1/2}\left(\int_a^b (b-t)\int_t^b |g'(s)|^2\,ds\,dt\right)^{1/2}.$$

By the computations in the proofs of Theorems 2.1 and 2.4, we have

$$\int_a^b (t-a)\int_a^t |f'(s)|^2\,ds\,dt = \int_a^b w_a(t;a,b)\,|f'(t)|^2\,dt$$

and

$$\int_a^b (b-t)\int_t^b |g'(s)|^2\, ds\, dt = \int_a^b w_b(t;a,b)\,|g'(t)|^2\, dt.$$

Thus,

$$\int_a^b |f(t)g(t)|\, dt \le \left(\int_a^b w_a(t;a,b)\,|f'(t)|^2\, dt\right)^{1/2}\left(\int_a^b w_b(t;a,b)\,|g'(t)|^2\, dt\right)^{1/2},$$

which proves the first inequality in (2.12).

Using (2.5), we obtain

$$\begin{aligned}
&\left(\int_a^b w_a(t;a,b)\,|f'(t)|^2\, dt\right)^{1/2}\left(\int_a^b w_b(t;a,b)\,|g'(t)|^2\, dt\right)^{1/2}\\
&\le \frac{1}{2}\left[\int_a^b w_a(t;a,b)\,|f'(t)|^2\, dt + \int_a^b w_b(t;a,b)\,|g'(t)|^2\, dt\right]\\
&= \frac{1}{4}(b-a)^2\int_a^b \left(|f'(t)|^2+|g'(t)|^2\right) dt - \frac{1}{4}\int_a^b \left((t-a)^2|f'(t)|^2+(b-t)^2|g'(t)|^2\right) dt,
\end{aligned}$$

which proves the second inequality in (2.12). □

Remark 2.8. Assume that f' is absolutely continuous on $[a,b]$. If $f(a)=0$, $f'(b)=0$, $f'\in L_{2,w_a}[a,b]$, and $f''\in L_{2,w_b}[a,b]$, then

$$\begin{aligned}
\int_a^b \left|f(t)\,f'(t)\right| dt &\le \left(\int_a^b w_a(t;a,b)\left|f'(t)\right|^2 dt\right)^{1/2}\left(\int_a^b w_b(t;a,b)\left|f''(t)\right|^2 dt\right)^{1/2}\\
&\le \frac{1}{4}(b-a)^2\int_a^b \left(\left|f'(t)\right|^2+\left|f''(t)\right|^2\right) dt\\
&\quad - \frac{1}{4}\int_a^b \left((t-a)^2\left|f'(t)\right|^2+(b-t)^2\left|f''(t)\right|^2\right) dt.
\end{aligned}$$

Corollary 2.9. *Let $f\colon [a,b]\to\mathbb{C}$ be absolutely continuous. Assume that*

$$f(a)=f(b)=0,\qquad f'\in L_{2,w_a}[a,b]\cap L_{2,w_b}[a,b].$$

Then

$$\begin{aligned}
\int_a^b \left|f(t)\right|^2 dt &\le \left(\int_a^b w_a(t;a,b)\left|f'(t)\right|^2 dt\right)^{1/2}\left(\int_a^b w_b(t;a,b)\left|f'(t)\right|^2 dt\right)^{1/2} \qquad (2.13)\\
&\le \int_a^b \left[\frac{1}{4}(b-a)^2-\left(t-\frac{a+b}{2}\right)^2\right]\left|f'(t)\right|^2 dt\\
&= \int_a^b (b-t)(t-a)\left|f'(t)\right|^2 dt.
\end{aligned}$$

Moreover, the inequalities in (2.13) are sharp.

Proof. From (2.12) with $g=f$ we obtain

$$\begin{aligned}
\int_a^b |f(t)|^2\, dt &\le \left(\int_a^b w_a(t;a,b)\,|f'(t)|^2\, dt\right)^{1/2}\left(\int_a^b w_b(t;a,b)\,|f'(t)|^2\, dt\right)^{1/2}\\
&\le \frac{1}{2}(b-a)^2\int_a^b |f'(t)|^2\, dt - \frac{1}{2}\int_a^b \left[(t-a)^2+(b-t)^2\right]|f'(t)|^2\, dt.
\end{aligned}$$

Since

$$\frac{(t-a)^2+(b-t)^2}{2}=\frac{1}{4}(b-a)^2+\left(t-\frac{a+b}{2}\right)^2,$$

we have

$$\begin{aligned}\frac{1}{2}\int_a^b\left[(t-a)^2+(b-t)^2\right]|f'(t)|^2\,dt&=\int_a^b\left[\frac{1}{4}(b-a)^2+\left(t-\frac{a+b}{2}\right)^2\right]|f'(t)|^2\,dt\\&=\frac{1}{4}(b-a)^2\int_a^b|f'(t)|^2\,dt+\int_a^b\left(t-\frac{a+b}{2}\right)^2|f'(t)|^2\,dt.\end{aligned}$$

Therefore,

$$\begin{aligned}&\frac{1}{2}(b-a)^2\int_a^b|f'(t)|^2\,dt-\frac{1}{2}\int_a^b\left[(t-a)^2+(b-t)^2\right]|f'(t)|^2\,dt\\&=\int_a^b\left[\frac{1}{4}(b-a)^2-\left(t-\frac{a+b}{2}\right)^2\right]|f'(t)|^2\,dt,\end{aligned}$$

which proves the second inequality in (2.13).

To show sharpness, consider

$$f(t)=(t-a)(b-t),\qquad t\in[a,b].$$

Then $f'(t)=a+b-2t$ for $t\in(a,b)$, and

$$\int_a^b|f(t)|^2\,dt=\int_a^b(t-a)^2(b-t)^2\,dt=\frac{1}{30}(b-a)^5.$$

Moreover, since $a+b-2t=-2\left(t-\frac{a+b}{2}\right)$, we obtain

$$\int_a^b(b-t)(t-a)\,|f'(t)|^2\,dt=4\int_a^b(b-t)(t-a)\left(t-\frac{a+b}{2}\right)^2\,dt=\frac{1}{30}(b-a)^5.$$

Thus equality holds in (2.13), and the inequalities in (2.13) are sharp. □

Theorem 2.10. *Let $f,g\colon[a,b]\to\mathbb{C}$ be absolutely continuous. Assume that*

$$f(a)=g(a)=f(b)=g(b)=0,\qquad f',\,g'\in L_{2,w_a}[a,b]\cap L_{2,w_b}[a,b].$$

Then

$$\int_a^b\left|f(t)\,g(t)\right|dt\le\frac{1}{4}\int_a^b L(t;a,b)\left(\left|f'(t)\right|^2+\left|g'(t)\right|^2\right)\,dt,$$

where

$$L(t;a,b):=\frac{1}{4}(b-a)^2-\begin{cases}(t-a)^2, & t\in\left[a,\frac{a+b}{2}\right],\\(b-t)^2, & t\in\left(\frac{a+b}{2},b\right].\end{cases}\tag{2.14}$$

Proof. From (2.1) we obtain

$$\begin{aligned}\int_a^{\frac{a+b}{2}}\left|f(t)g(t)\right|dt&\le\frac{1}{2}\int_a^{\frac{a+b}{2}}w_a\!\left(t;a,\frac{a+b}{2}\right)\left(|f'(t)|^2+|g'(t)|^2\right)\,dt\\&=\frac{1}{4}\int_a^{\frac{a+b}{2}}\left[\left(\frac{a+b}{2}-a\right)^2-(t-a)^2\right]\left(|f'(t)|^2+|g'(t)|^2\right)\,dt.\end{aligned}\tag{2.15}$$

Similarly, from (2.7) we have

$$\int_{\frac{a+b}{2}}^{b}\left|f(t)g(t)\right|dt \le \frac{1}{2}\int_{\frac{a+b}{2}}^{b} w_b\left(t;\frac{a+b}{2},b\right)\left(|f'(t)|^2+|g'(t)|^2\right)dt$$
$$= \frac{1}{4}\int_{\frac{a+b}{2}}^{b}\left[\left(b-\frac{a+b}{2}\right)^2-(b-t)^2\right]\left(|f'(t)|^2+|g'(t)|^2\right)dt. \tag{2.16}$$

Adding (2.15) and (2.16), we obtain

$$\begin{aligned}
&\int_a^b \left|f(t)g(t)\right|dt \\
&\le \frac{1}{4}\int_a^{\frac{a+b}{2}}\left[\frac{1}{4}(b-a)^2-(t-a)^2\right]\left(|f'(t)|^2+|g'(t)|^2\right)dt \\
&\quad + \frac{1}{4}\int_{\frac{a+b}{2}}^{b}\left[\frac{1}{4}(b-a)^2-(b-t)^2\right]\left(|f'(t)|^2+|g'(t)|^2\right)dt \\
&= \frac{1}{16}(b-a)^2\int_a^{\frac{a+b}{2}}\left(|f'(t)|^2+|g'(t)|^2\right)dt + \frac{1}{16}(b-a)^2\int_{\frac{a+b}{2}}^{b}\left(|f'(t)|^2+|g'(t)|^2\right)dt \\
&\quad - \frac{1}{4}\int_a^{\frac{a+b}{2}}(t-a)^2\left(|f'(t)|^2+|g'(t)|^2\right)dt - \frac{1}{4}\int_{\frac{a+b}{2}}^{b}(b-t)^2\left(|f'(t)|^2+|g'(t)|^2\right)dt.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int_a^b \left|f(t)g(t)\right|dt &= \frac{1}{16}(b-a)^2\int_a^b\left(|f'(t)|^2+|g'(t)|^2\right)dt - \frac{1}{4}\int_a^{\frac{a+b}{2}}(t-a)^2\left(|f'(t)|^2+|g'(t)|^2\right)dt \\
&\quad - \frac{1}{4}\int_{\frac{a+b}{2}}^{b}(b-t)^2\left(|f'(t)|^2+|g'(t)|^2\right)dt \\
&= \frac{1}{4}\int_a^b L(t;a,b)\left(|f'(t)|^2+|g'(t)|^2\right)dt,
\end{aligned}$$

which proves the desired result. □

Corollary 2.11. *Let $f\colon [a,b]\to\mathbb{C}$ be absolutely continuous. Assume that*

$$f(a)=f(b)=0, \qquad f'\in L_{2,w_a}[a,b]\cap L_{2,w_b}[a,b].$$

Then

$$\int_a^b |f(t)|^2\,dt \le \frac{1}{2}\int_a^b L(t;a,b)\,|f'(t)|^2\,dt, \tag{2.17}$$

where $L(t;a,b)$ is defined in (2.14). Moreover, the constant $\frac{1}{2}$ is best possible.

Proof. Consider the function

$$f(t) := \begin{cases} t-a, & t\in\left[a,\frac{a+b}{2}\right], \\ b-t, & t\in\left(\frac{a+b}{2},b\right]. \end{cases}$$

Then f is absolutely continuous and $|f'(t)| = 1$ for $t \in (a, b)$. Moreover,

$$\int_a^b |f(t)|^2\,dt = \int_a^{\frac{a+b}{2}} (t-a)^2\,dt + \int_{\frac{a+b}{2}}^b (b-t)^2\,dt$$
$$= \frac{1}{24}(b-a)^3 + \frac{1}{24}(b-a)^3 = \frac{1}{12}(b-a)^3.$$

On the other hand, since $|f'(t)| = 1$,

$$\frac{1}{2}\int_a^b L(t;a,b)\,|f'(t)|^2\,dt = \frac{1}{2}\int_a^b L(t;a,b)\,dt$$
$$= \frac{1}{2}\int_a^b \frac{1}{4}(b-a)^2\,dt - \frac{1}{2}\left[\int_a^{\frac{a+b}{2}} (t-a)^2\,dt + \int_{\frac{a+b}{2}}^b (b-t)^2\,dt\right]$$
$$= \frac{1}{8}(b-a)^3 - \frac{1}{2}\cdot\frac{1}{12}(b-a)^3 = \frac{1}{12}(b-a)^3.$$

Thus both sides of (2.17) are equal to $\frac{1}{12}(b-a)^3$ for this choice of f. This proves that the constant $\frac{1}{2}$ is best possible. □

Remark 2.12. Assume that f' is absolutely continuous on $[a, b]$. If $f(a) = f'(a) = 0$, $f(b) = f'(b) = 0$, and

$$f',\ f'' \in L_{2,w_a}[a,b] \cap L_{2,w_b}[a,b],$$

then

$$\int_a^b \left|f(t)\,f'(t)\right| dt \le \frac{1}{4}\int_a^b L(t;a,b)\left(\left|f'(t)\right|^2 + \left|f''(t)\right|^2\right) dt.$$

3 Applications

We have the following trapezoid-type inequalities.

Proposition 3.1. *Let $g \in C^1([a,b],\mathbb{C})$. Then*

$$\left|\frac{g(a)+g(b)}{2} - \frac{1}{b-a}\int_a^b g(t)\,dt\right|^2 \tag{3.1}$$
$$\le \frac{1}{4}\left(\frac{1}{b-a}\int_a^b w_a(t;a,b)\left|g'(t)-g'(a+b-t)\right|^2 dt\right)^{1/2}$$
$$\times\left(\frac{1}{b-a}\int_a^b w_b(t;a,b)\left|g'(t)-g'(a+b-t)\right|^2 dt\right)^{1/2}$$
$$\le \frac{1}{4}\,\frac{1}{b-a}\int_a^b (b-t)(t-a)\left|g'(t)-g'(a+b-t)\right|^2 dt.$$

Proof. Let $g \in C^1([a,b],\mathbb{C})$ and define

$$f(t) := \frac{g(t)+g(a+b-t)}{2} - \frac{g(a)+g(b)}{2}, \qquad t \in [a,b].$$

Then $f(a) = f(b) = 0$, and

$$f'(t) = \frac{g'(t)-g'(a+b-t)}{2}, \qquad t \in (a,b).$$

Applying (2.13) to f, we obtain

$$\int_a^b \left| \frac{g(t)+g(a+b-t)}{2} - \frac{g(a)+g(b)}{2} \right|^2 dt \tag{3.2}$$
$$\le \frac{1}{4}\left(\int_a^b w_a(t;a,b)\left|g'(t)-g'(a+b-t)\right|^2 dt\right)^{1/2}\left(\int_a^b w_b(t;a,b)\left|g'(t)-g'(a+b-t)\right|^2 dt\right)^{1/2}$$
$$\le \frac{1}{4}\int_a^b (b-t)(t-a)\left|g'(t)-g'(a+b-t)\right|^2 dt.$$

By the Cauchy–Bunyakovsky–Schwarz inequality,

$$(b-a)\int_a^b \left| \frac{g(t)+g(a+b-t)}{2} - \frac{g(a)+g(b)}{2} \right|^2 dt \ge \left| \int_a^b \left[\frac{g(t)+g(a+b-t)}{2} - \frac{g(a)+g(b)}{2} \right] dt \right|^2$$
$$= \left| \int_a^b g(t)\,dt - \frac{g(a)+g(b)}{2}(b-a) \right|^2 .$$

Hence

$$\left| \frac{g(a)+g(b)}{2} - \frac{1}{b-a}\int_a^b g(t)\,dt \right|^2 \le \frac{1}{b-a}\int_a^b \left| \frac{g(t)+g(a+b-t)}{2} - \frac{g(a)+g(b)}{2} \right|^2 dt. \tag{3.3}$$

Combining (3.2) and (3.3) yields (3.1). □

From a different perspective, we also have the following result.

Proposition 3.2. *Let $g \in C^1([a,b],\mathbb{C})$. Then*

$$\left| \frac{g(a)+g(b)}{2} - \frac{1}{b-a}\int_a^b g(t)\,dt \right|^2 \tag{3.4}$$
$$\le \left(\frac{1}{b-a}\int_a^b w_a(t;a,b)\left| g'(t) - \frac{g(b)-g(a)}{b-a} \right|^2 dt \right)^{1/2}$$
$$\times \left(\frac{1}{b-a}\int_a^b w_b(t;a,b)\left| g'(t) - \frac{g(b)-g(a)}{b-a} \right|^2 dt \right)^{1/2}$$
$$\le \frac{1}{b-a}\int_a^b (b-t)(t-a)\left| g'(t) - \frac{g(b)-g(a)}{b-a} \right|^2 dt.$$

Proof. Let $g \in C^1([a,b],\mathbb{C})$ and define

$$f(t) := g(t) - \frac{g(a)(b-t)+g(b)(t-a)}{b-a}, \qquad t \in [a,b].$$

Then $f(a) = f(b) = 0$ and

$$f'(t) = g'(t) - \frac{g(b)-g(a)}{b-a}, \qquad t \in (a,b).$$

Applying (2.13) to f, we obtain

$$\int_a^b \left| g(t) - \frac{g(a)(b-t) + g(b)(t-a)}{b-a} \right|^2 dt \tag{3.5}$$
$$\leq \left(\int_a^b w_a(t;a,b) \left| g'(t) - \frac{g(b) - g(a)}{b-a} \right|^2 dt \right)^{1/2} \left(\int_a^b w_b(t;a,b) \left| g'(t) - \frac{g(b) - g(a)}{b-a} \right|^2 dt \right)^{1/2}$$
$$\leq \int_a^b (b-t)(t-a) \left| g'(t) - \frac{g(b) - g(a)}{b-a} \right|^2 dt.$$

By the Cauchy–Bunyakovsky–Schwarz inequality, we also have

$$\left| \frac{g(a) + g(b)}{2} - \frac{1}{b-a} \int_a^b g(t)\, dt \right|^2 \leq \frac{1}{b-a} \int_a^b \left| g(t) - \frac{g(a)(b-t) + g(b)(t-a)}{b-a} \right|^2 dt. \tag{3.6}$$

Combining (3.5) and (3.6) yields (3.4). □

We also have the following result.

Proposition 3.3. *Let $g \in C([a,b], \mathbb{C})$. Then*

$$\left| \frac{a+b}{2} \frac{1}{b-a} \int_a^b g(s)\, ds - \frac{1}{b-a} \int_a^b t\, g(t)\, dt \right|^2 \tag{3.7}$$
$$\leq \left(\frac{1}{b-a} \int_a^b w_a(t;a,b) \left| g(t) - \frac{1}{b-a} \int_a^b g(s)\, ds \right|^2 dt \right)^{1/2}$$
$$\times \left(\frac{1}{b-a} \int_a^b w_b(t;a,b) \left| g(t) - \frac{1}{b-a} \int_a^b g(s)\, ds \right|^2 dt \right)^{1/2}$$
$$\leq \frac{1}{b-a} \int_a^b (b-t)(t-a) \left| g(t) - \frac{1}{b-a} \int_a^b g(s)\, ds \right|^2 dt.$$

Proof. Let $g \in C([a,b], \mathbb{C})$ and define

$$f(t) := \int_a^t g(s)\, ds - \frac{t-a}{b-a} \int_a^b g(s)\, ds, \qquad t \in [a,b].$$

Then $f(a) = f(b) = 0$ and

$$f'(t) = g(t) - \frac{1}{b-a} \int_a^b g(s)\, ds, \qquad t \in (a,b).$$

Applying (2.13) to f, we obtain

$$\int_a^b \left| \int_a^t g(s)\, ds - \frac{t-a}{b-a} \int_a^b g(s)\, ds \right|^2 dt \tag{3.8}$$
$$\leq \left(\int_a^b w_a(t;a,b) \left| g(t) - \frac{1}{b-a} \int_a^b g(s)\, ds \right|^2 dt \right)^{1/2}$$
$$\times \left(\int_a^b w_b(t;a,b) \left| g(t) - \frac{1}{b-a} \int_a^b g(s)\, ds \right|^2 dt \right)^{1/2}$$
$$\leq \int_a^b (b-t)(t-a) \left| g(t) - \frac{1}{b-a} \int_a^b g(s)\, ds \right|^2 dt.$$

Moreover, integrating by parts yields

$$\begin{aligned}\int_a^b \left(\int_a^t g(s)\,ds - \frac{t-a}{b-a}\int_a^b g(s)\,ds\right) dt &= \int_a^b \left(\int_a^t g(s)\,ds\right) dt - \frac{b-a}{2}\int_a^b g(s)\,ds \\ &= \left[t\int_a^t g(s)\,ds\right]_{t=a}^{t=b} - \int_a^b t\,g(t)\,dt - \frac{b-a}{2}\int_a^b g(s)\,ds \\ &= b\int_a^b g(s)\,ds - \int_a^b t\,g(t)\,dt - \frac{b-a}{2}\int_a^b g(s)\,ds \\ &= \frac{a+b}{2}\int_a^b g(s)\,ds - \int_a^b t\,g(t)\,dt.\end{aligned}$$

By the Cauchy–Bunyakovsky–Schwarz inequality,

$$\begin{aligned}(b-a)\int_a^b \left|\int_a^t g(s)\,ds - \frac{t-a}{b-a}\int_a^b g(s)\,ds\right|^2 dt &\geq \left|\int_a^b \left(\int_a^t g(s)\,ds - \frac{t-a}{b-a}\int_a^b g(s)\,ds\right) dt\right|^2 \\ &= \left|\frac{a+b}{2}\int_a^b g(s)\,ds - \int_a^b t\,g(t)\,dt\right|^2.\end{aligned} \tag{3.9}$$

Combining (3.8) and (3.9) yields (3.7). □

Let us now consider the weighted Čebyšev functional

$$C_w(f,g) := \int_a^b w(t)\,f(t)\,g(t)\,dt - \left(\int_a^b w(t)\,f(t)\,dt\right)\left(\int_a^b w(t)\,g(t)\,dt\right),$$

where $f, g : [a,b] \to \mathbb{C}$ and $w : [a,b] \to (0,\infty)$ is continuous on $[a,b]$ and satisfies

$$\int_a^b w(t)\,dt = 1.$$

Theorem 3.4. *Let $f \in L_2([a,b],\mathbb{C})$ and let $g \in C^1([a,b],\mathbb{C})$. Then*

$$\begin{aligned}&\left|C_w(f,g)\right|^2 \\ &\leq \left(\int_a^b |g'(t)|^2\,dt\right)\left(\int_a^b w_a(t;a,b)\left|f(t) - \int_a^b f(s)\,w(s)\,ds\right|^2 w^2(t)\,dt\right)^{1/2} \\ &\quad \times \left(\int_a^b w_b(t;a,b)\left|f(t) - \int_a^b f(s)\,w(s)\,ds\right|^2 w^2(t)\,dt\right)^{1/2} \\ &\leq \left(\int_a^b |g'(t)|^2\,dt\right)\int_a^b (b-t)(t-a)\left|f(t) - \int_a^b f(s)\,w(s)\,ds\right|^2 w^2(t)\,dt \\ &\leq \frac{1}{4}(b-a)^2\left(\int_a^b |g'(t)|^2\,dt\right)\int_a^b \left|f(t) - \int_a^b f(s)\,w(s)\,ds\right|^2 w^2(t)\,dt.\end{aligned} \tag{3.10}$$

Proof. Integrating by parts, we have

$$\begin{aligned}
&\int_a^b \left(\int_a^x f(t)w(t)\,dt - \int_a^x w(s)\,ds \int_a^b f(s)w(s)\,ds\right) g'(x)\,dx \\
&= \left[\left(\int_a^x f(t)w(t)\,dt - \int_a^x w(s)\,ds \int_a^b f(s)w(s)\,ds\right) g(x)\right]_{x=a}^{x=b} \\
&\quad - \int_a^b g(x)\left(f(x)w(x) - w(x)\int_a^b f(s)w(s)\,ds\right) dx \\
&= -\int_a^b f(x)g(x)w(x)\,dx + \left(\int_a^b f(s)w(s)\,ds\right)\left(\int_a^b g(x)w(x)\,dx\right).
\end{aligned}$$

Consequently,

$$C_w(f,g) = \int_a^b \left(\int_a^x w(s)\,ds \int_a^b f(s)w(s)\,ds - \int_a^x f(t)w(t)\,dt\right) g'(x)\,dx.$$

Using the Cauchy–Bunyakovsky–Schwarz inequality, we obtain

$$\begin{aligned}
|C_w(f,g)|^2 &= \left|\int_a^b \left(\int_a^x w(s)\,ds \int_a^b f(s)w(s)\,ds - \int_a^x f(t)w(t)\,dt\right) g'(x)\,dx\right|^2 \\
&\le \left(\int_a^b \left|\int_a^x w(s)\,ds \int_a^b f(s)w(s)\,ds - \int_a^x f(t)w(t)\,dt\right|^2 dx\right)\left(\int_a^b |g'(x)|^2 dx\right).
\end{aligned} \tag{3.11}$$

Now take, for $x \in [a,b]$,

$$h(x) := \int_a^x w(s)\,ds \int_a^b f(s)w(s)\,ds - \int_a^x f(t)w(t)\,dt.$$

Then $h(a) = 0$ and, since $\int_a^b w(s)\,ds = 1$, we also have $h(b) = 0$. Moreover, h is absolutely continuous on $[a,b]$ and

$$h'(x) = w(x)\left(\int_a^b f(s)w(s)\,ds - f(x)\right) \qquad \text{for a.e. } x \in (a,b).$$

Applying (2.13) to h, we get

$$\begin{aligned}
\int_a^b |h(t)|^2\,dt &\le \left(\int_a^b w_a(t;a,b)\,|h'(t)|^2\,dt\right)^{1/2}\left(\int_a^b w_b(t;a,b)\,|h'(t)|^2\,dt\right)^{1/2} \\
&\le \int_a^b (b-t)(t-a)\,|h'(t)|^2\,dt.
\end{aligned} \tag{3.12}$$

Since

$$|h'(t)|^2 = \left|f(t) - \int_a^b f(s)w(s)\,ds\right|^2 w^2(t),$$

by (3.11) and (3.12) we obtain the inequalities in (3.10). The last bound in (3.10) follows from

$$(b-t)(t-a) \le \frac{1}{4}(b-a)^2, \qquad t \in [a,b].$$

□

Conflict of Interest

The author declares no conflict of interest.

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