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A Fuzzy Counterpart of Suzuki's Fixed Point Theorem That Involves Fuzzy w-Distances

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Abstract. In a recent article [Mathematics (2024), 2024:305] it was shown that the renowned Suzuki fixed point theorem [Proc. Amer. Math. Soc. 136 (2008), 1861–1869] cannot be generalized in an obvious and natural way to the setting of w-distances. In contrast to this situation, we here show that our fuzzy counterpart of Suzuki's theorem [J. Nonlinear Convex Anal. 23 (2024), 1487-1494] admits a full generalization to the framework of fuzzy w-distances.

Keywords: fixed point; fuzzy metric space; fuzzy w-distance; w-Suzuki fuzzy contraction
MSC (2020): 47H10, 54H25, 54A40

1 Introduction and preliminaries

In their influential article [7], Kada et al. introduced and discussed the notion of w-distance in the realm of metric spaces. They demonstrated that this structure provides an appealing context for refining and generalizing several important results such as the Ekeland variational principle [4], the Caristi fixed point theorem [3] and the Takahashi nonconvex minimization theorem [25], among others. In fact, the famous Kannan fixed point theorem also admits a satisfactory w-distance generalization [21, 22], while complete metric spaces can be characterized by means of a type of contractions that have a fixed point and involve w-distances [24]. These achievements motivated many authors to continue the research concerning this kind of distances from different approaches and viewpoints. In particular, the fixed point theory based on the use of w-distances received a strong boost (see, e.g., [1, 2, 8, 10, 12–14] and the monograph [15] with the references therein).

In what follows, by $\mathbb{R}^+$, $\mathbb{N}$ and $\mathbb{N}_0$ we will denote the set of all non-negative real numbers, the set of all natural numbers and the set of all non-negative integer numbers, respectively.

Let us recall [7] that a w-distance on a metric space (X, d) is a function $w : X \times X \to \mathbb{R}^+$ that satisfies the following conditions:

(w1) $w(x, y) \leq w(x, z) + w(z, y)$, for all $x, y, z \in X$;

(w2) for each $x \in X$, the function $w(x, \_) : X \to \mathbb{R}^+$ is lower semicontinuous;

(w3) for each $\varepsilon > 0$ there exists $\delta > 0$ such that $w(x, y) \leq \delta$ and $w(x, z) \leq \delta$ imply $d(y, z) \leq \varepsilon$.

Opportune examples of w-distances may be found in [7, 15, 22, 24]. In particular, every metric d on a set X is a w-distance on the metric space (X, d).

On the other hand, Suzuki showed in his prominent article [23] a generalization of the Banach contraction principle that allows to obtain a characterization of the metric completeness. With the aim of simplifying its presentation, we state Suzuki's theorem as follows.

Theorem 1.1. (Suzuki fixed point theorem) *Let (X, d) be a complete metric space. If T is a self-mapping of X such that there exists a constant $\alpha \in (0, 1)$ satisfying the condition*

$$d(x, Tx) \leq 2d(x, y) \Longrightarrow d(Tx, Ty) \leq \alpha d(x, y), \tag{1.1}$$

for all $x, y \in X$, then, T has a unique fixed point.

Since the possible extension of Suzuki's theorem to the w-distance framework did not seem to have been explored yet, Romaguera and Tirado discussed this question in [18]. Specifically, they presented an example (see also [19, Remark 3.1]) of a self-mapping T on a complete metric space (X, d) such that T is free of fixed points but there exist a constant $\alpha \in (0, 1)$ and a w-distance w on (X, d) fulfilling the following contraction condition

$$w(x, Tx) \leq 2w(x, y) \Longrightarrow w(Tx, Ty) \leq \alpha w(x, y), \tag{1.2}$$

for all $x, y \in X$.

Then, they introduced the notion of presymmetric w-distance and proved that if T is a self-mapping on a complete metric space (X, d) for which there exist a constant $\alpha \in (0, 1)$ and a presymmetric w-distance w on (X, d) such that the contraction condition (1.2) is satisfied, then T has a unique fixed point. In fact, Suzuki's theorem is a consequence of this result.

In Section 2 of this paper we show that the situation is very different in the fuzzy setting. Specifically, we prove that the fuzzy counterpart of Suzuki's theorem obtained in [16] (see Theorem 1.3 below) admits a full generalization to the w-distance framework without adding extra conditions. Two illustrative examples are also given.

In order to help the reader, we next recall several pertinent concepts and properties concerning fuzzy metric spaces in the sense of Kramosil and Michálek [11] (our notation and terminology are standard).

According to [5,9] a continuous t-norm is a binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ such that $([0, 1], \leq, *)$ is an ordered Abelian topological monoid with unit 1. Typical and useful examples of continuous t-norm are $\wedge$, $*_P$ and $*_L$ (the Łukasiewicz t-norm), which are defined as follows: $u \wedge v = \min\{u, v\}$, $u *_P v = uv$, and $u *_L v = \max\{u + v - 1, 0\}$, for all $u, v \in [0, 1]$. Besides, we have $\wedge \geq *$ for any continuous t-norm $*$.

A fuzzy metric on a (non-empty) set X is a pair $(M, *)$ such that M is a fuzzy set in $X \times X \times \mathbb{R}^+$ and $*$ is a continuous t-norm satisfying the following conditions for all $x, y, z \in X$:

(FM1) $M(x, y, 0) = 0$;

(FM2) $x = y$ if and only if $M(x, y, t) = 1$ for all $t > 0$;

(FM3) $M(x, y, t) = M(y, x, t)$ for all $t > 0$;

(FM4) $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$ for all $t, s \in \mathbb{R}^+$;

(FM5) $M(x, y, \_) : \mathbb{R}^+ \to [0, 1]$ is a left continuous function.

A fuzzy metric space is a 3-tuple $(X, M, *)$ where X is a (non-empty) set and $(M, *)$ is a fuzzy metric on X.

It is clear that for each $x, y \in X$, $M(x, y, \_)$ is a non-decreasing function on $\mathbb{R}^+$. Moreover, condition (FM2) can be formulated in a more general form as: $x = y$ if and only if $M(x, y, t) > 1 - t$ for all $t > 0$.

Each fuzzy metric $(M, *)$ on a set X induces a topology τ_M which has as a base of open sets the family $\{B_M(x, \varepsilon, t) : \varepsilon \in (0, 1), t > 0\}$, where $B_M(x, \varepsilon, t) = \{y \in X : M(x, y, t) > 1 - \varepsilon\}$. Therefore, a sequence $(x_n)_{n\in\mathbb{N}}$ is τ_M-convergent to $x \in X$ if and only if for each $t > 0$, $M(x, x_n, t) \to 1$ as $n \to \infty$. We shall write $x_n \to x$ whenever that the sequence $(x_n)_{n\in\mathbb{N}}$ τ_M-converges to $x \in X$ and no confusion arises.

A sequence $(x_n)_{n\in\mathbb{N}}$ in a fuzzy metric space $(X, M, *)$ is called a Cauchy sequence if for each $\varepsilon \in (0, 1)$ and $t > 0$ there exists an $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for all $n, m \geq n_0$. $(X, M, *)$ is said to be complete if every Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ is τ_M-convergent in X.

It is well known that every fuzzy metric space is metrizable, i.e., given a fuzzy metric space $(X, M, *)$ there exists a metric on X whose induced topology coincides with τ_M.

The following typical example of a fuzzy metric space will be useful later on.

Example 1.2. Let (X, d) be a metric space. For each $x, y \in X$ and $t \in \mathbb{R}^+$ define $M(x, y, t) = 0$ if $d(x, y) \geq t$, and $M(x, y, t) = 1$ if $d(x, y) < t$. It is well known that, for each continuous t-norm $*$, the 3-tuple $(X, M, *)$ is a fuzzy metric space such that τ_M agrees with the topology induced by d. Furthermore, $(X, M, *)$ is complete if and only if (X, d) is complete.

Generalizing the concept of a C-contraction introduced by Hicks [6], it was proposed in [17] the following notion:

A Suzuki fuzzy contraction on a fuzzy metric space $(X, M, *)$ is a self-mapping T of X for which there exists a constant $\alpha \in (0, 1)$ such that for each $x, y \in X$ and $t > 0$,

$$(x, y) \in S(t) \Longrightarrow M(Tx, Ty, \alpha t) > 1 - \alpha t,$$

where

$$S(t) := \{(x, y) \in X \times X : \min\{M(x, y, t), M(x, Tx, t)\} > 1 - t\}.$$

Then, it was proved the following counterpart of Theorem 1.1.

Theorem 1.3. ([17]) *Every Suzuki fuzzy contraction on a complete fuzzy metric space has a unique fixed point.*

In Section 2 we shall generalize the preceding theorem to fuzzy w-distances, as we indicated above.

2 Main Result and Examples

The concept of w-distance was extended to the field of fuzzy metric spaces in [16] as follows (see [20] for a previous notion, introduced under the name of a r-distance).

A fuzzy w-distance on a fuzzy metric space $(X, M, *)$ is a fuzzy set W in $X \times X \times \mathbb{R}^+$ that satisfies the next conditions for all $x, y, z \in X$:

(FW1) $W(x, y, s + t) \geq W(x, z, s) * W(z, y, t)$ for all $s, t \in \mathbb{R}^+$;

(FW2) if $x \in X$ and $y_n \to y$, then $W(x, y, t + \varepsilon) \geq \limsup_n W(x, y_n, t)$, for all $t > 0$ and $\varepsilon \in (0, t)$;

(FW3) for each $\varepsilon \in (0, 1)$ there is $\delta \in (0, 1)$ such that $W(x, y, s) \geq 1 - \delta$ and $W(x, z, t) \geq 1 - \delta$ imply $M(y, z, s + t) \geq 1 - \varepsilon$.

Note that it follows from (FW2) that $W(x, y, t) \geq W(x, y, s)$ for all $x, y \in X$ and $t > s > 0$.

Several examples of fuzzy w-distances may be found, for instance, in [16, 20]. In particular, every fuzzy metric $(M, *)$ on a set X is a fuzzy w-distance on the fuzzy metric space $(X, M, *)$.

By a w-Suzuki fuzzy contraction on a fuzzy metric space $(X, M, *)$ we mean a self-mapping T of X for which there exist a fuzzy w-distance W on $(X, M, *)$ and a constant $\alpha \in (0, 1)$ such that

$$(x, y) \in WS(t) \Longrightarrow W(Tx, Ty, \alpha t) > 1 - \alpha t,$$

for all $x, y \in X$ and $t > 0$, where

$$WS(t) := \{(x, y) \in X \times X : \min\{W(x, y, t), W(x, Tx, t)\} > 1 - t\}.$$

Next we state and prove our main result.

Theorem 2.1. *Every w-Suzuki fuzzy contraction on a complete fuzzy metric space has a unique fixed point.*

Proof. Let $(X, M, *)$ be a complete fuzzy metric space and let T be a w-Suzuki fuzzy contraction on $(X, M, *)$ Then, there exist a fuzzy w-distance W on $(X, M, *)$ and a constant $\alpha \in (0, 1)$ such that

$$(x, y) \in WS(t) \Longrightarrow W(Tx, Ty, \alpha t) > 1 - \alpha t, \tag{2.1}$$

for all $x, y \in X$ and $t > 0$, where

$$WS(t) := \{(x, y) \in X \times X : \min\{W(x, y, t), W(x, Tx, t)\} > 1 - t\}.$$

Fix $t_0 > 1$. For any $x, y \in X$ we have $W(x, y, t_0) > 1 - t_0$ and $W(x, Tx, t_0) > 1 - t_0$, so $(x, y) \in WS(t_0)$, and, hence,

$$W(Tx, Ty, \alpha t_0) > 1 - \alpha t_0, \tag{2.2}$$

by (2.1). Note that, in particular, $(x, Tx) \in WS(t_0)$, and, thus,

$$W(Tx, T^2x, \alpha t_0) > 1 - \alpha t_0. \tag{2.3}$$

From (2.2) and (2.3) we deduce that $(Tx, Ty) \in WS(\alpha t_0)$, so, by (2.1),

$$W(T^2x, T^2y, \alpha^2 t_0) > 1 - \alpha^2 t_0.$$

Repeating this process we get that

$$(T^n x, T^n y) \in WS(\alpha^n t_0), \tag{2.4}$$

and

$$W(T^n x, T^n y, \alpha^n t_0) > 1 - \alpha^n t_0, \tag{2.5}$$

for all $x, y \in X$ and $n \in \mathbb{N}_0$.

Now, fix $x_0 \in X$ and put $x_n := T^n x_0$ for all $n \in \mathbb{N}_0$. We shall show that $\{x_n\}_{n\in\mathbb{N}_0}$ is a Cauchy sequence in $(X, M, *)$. Indeed, choose an arbitrary $\varepsilon \in (0, 1)$. By condition (FW3), there exists $\delta \in (0, 1)$ such that $M(y, z, s + t) \geq 1 - \varepsilon/2$ whenever $W(x, y, s) \geq 1 - \delta$ and $W(x, z, t) \geq 1 - \delta$, with $x, y, z \in X$ and $s, t \in \mathbb{R}^+$. Pick $k := k(\varepsilon) \in \mathbb{N}$ fulfilling $\alpha^k t_0 < \min\{\varepsilon/2, \delta\}$. By (2.5) we have

$$W(T^k x_0, T^k T^{n-k} x_0, \alpha^k t_0) > 1 - \alpha^k t_0 \quad \text{and} \quad W(T^k x_0, T^k T^{m-k} x_0, \alpha^k t_0) > 1 - \alpha^k t_0,$$

whenever $m > n > k$, i.e.,

$$W(x_k, x_n, \alpha^k t_0) > 1 - \alpha^k t_0 \quad \text{and} \quad W(x_k, x_m, \alpha^k t_0) > 1 - \alpha^k t_0, \tag{2.6}$$

whenever $m > n > k$, and, similarly,

$$W(x_{k+1}, x_n, \alpha^{k+1}t_0) > 1 - \alpha^{k+1}t_0 \quad \text{and} \quad W(x_{k+1}, x_m, \alpha^{k+1}t_0) > 1 - \alpha^{k+1}t_0, \tag{2.7}$$

whenever $m > n > k + 1$. Since $1 - \alpha^k t_0 > 1 - \delta$ it follows that

$$W(x_k, x_n, \alpha^k t_0) > 1 - \delta \quad \text{and} \quad W(x_k, x_m, \alpha^k t_0) > 1 - \delta,$$

so, by (FW3), with $s = t = \alpha^k t_0$,

$$M(x_n, x_m, 2\alpha^k t_0) \geq 1 - \frac{\varepsilon}{2},$$

and thus (recall that $\varepsilon > 2\alpha^k t_0$),

$$M(x_n, x_m, \varepsilon) \geq M(x_n, x_m, 2\alpha^k t_0) > 1 - \varepsilon,$$

whenever $m > n > k$. As ε is arbitrary, we conclude that $\{x_n\}_{n\in\mathbb{N}_0}$ is a Cauchy sequence in the complete fuzzy metric space $(X, M, *)$.

Hence, there is $z \in X$ such that $x_n \to z$, i.e., for each $t > 0$, $M(z, x_n, t) \to 1$ as $n \to \infty$.

Next, we are going to show that z is a fixed point of T. Indeed, for the $\varepsilon \in (0, 1)$ considered in the first part of the proof we can find $\eta \in (0, \alpha^{k+1}t_0)$ such that $\alpha^{k+1}t_0 + \eta < \min\{\varepsilon/2, \delta\}$. Then, by condition (FW2), we have

$$W(x_k, z, \alpha^k t_0 + \eta) \geq \limsup_n W(x_k, x_n, \alpha^k t_0),$$

and

$$W(x_{k+1}, z, \alpha^{k+1}t_0 + \eta) \geq \limsup_n W(x_{k+1}, x_n, \alpha^{k+1}t_0).$$

Let $i > k$ and $j > k + 1$ fulfilling respectively,

$$\eta + W(x_k, z, \alpha^k t_0 + \eta) > W(x_k, x_i, \alpha^k t_0), \tag{2.8}$$

and

$$\eta + W(x_{k+1}, z, \alpha^{k+1}t_0 + \eta) > W(x_{k+1}, x_j, \alpha^{k+1}t_0). \tag{2.9}$$

Therefore, by applying (2.6) and (2.8), we obtain

$$W(x_k, z, \alpha^k t_0 + \eta) > W(x_k, x_i, \alpha^k t_0) - \eta > 1 - (\alpha^k t_0 + \eta), \tag{2.10}$$

and by applying (2.7) and (2.9), we obtain

$$W(x_{k+1}, z, \alpha^{k+1}t_0 + \eta) > W(x_{k+1}, x_j, \alpha^{k+1}t_0) - \eta > 1 - (\alpha^{k+1}t_0 + \eta). \tag{2.11}$$

Furthermore, we get

$$W(x_k, x_{k+1}, \alpha^k t_0 + \eta) \geq W(x_k, x_{k+1}, \alpha^k t_0) > 1 - \alpha^k t_0 > 1 - (\alpha^k t_0 + \eta). \tag{2.12}$$

By (2.10) and (2.12) we deduce that $(x_k, z) \in WS(\alpha^k t_0 + \eta)$, which implies, by (2.1), that

$$W(x_{k+1}, Tz, \alpha(\alpha^k t_0 + \eta)) > 1 - \alpha(\alpha^k t_0 + \eta). \tag{2.13}$$

Since $1 - \alpha(\alpha^k t_0 + \eta) > 1 - (\alpha^{k+1}t_0 + \eta) > 1 - (\alpha^k t_0 + \eta) > 1 - \delta$, it follows from (2.11), (2.13) and condition (FW3) that

$$M(z, Tz, s + t) \geq 1 - \frac{\varepsilon}{2},$$

where $s = \alpha^{k+1}t_0 + \eta$ and $t = \alpha(\alpha^k t_0 + \eta)$. Note that $\varepsilon > s + t$ because $\alpha^k t_0 + \eta < \varepsilon/2$. Hence,

$$M(z, Tz, \varepsilon) \geq 1 - \varepsilon.$$

Since ε is arbitrary we have proved that z is a fixed point of T.

Finally, we check that z is the unique fixed point of T. Let $u \in X$ such that $u = Tu$. By (2.5) we have

$$W(z, z, \alpha^n t_0) > 1 - \alpha^n t_0 \quad \text{and} \quad W(z, u, \alpha^n t_0) > 1 - \alpha^n t_0$$

for all $n \in \mathbb{N}_0$. Choose an arbitrary $\varepsilon \in (0, 1)$. As in the first part of the proof, take $\delta \in (0, 1)$ such that $M(y, z, s + t) \geq 1 - \varepsilon/2$ whenever $W(x, y, s) \geq 1 - \delta$ and $W(x, z, t) \geq 1 - \delta$, with $x, y, z \in X$ and $s, t \in \mathbb{R}^+$, and take $k := k(\varepsilon) \in \mathbb{N}$ such that $\alpha^k t_0 < \min\{\varepsilon/2, \delta\}$. Then,

$$W(z, z, \alpha^k t_0) > 1 - \delta \quad \text{and} \quad W(z, u, \alpha^k t_0) > 1 - \delta,$$

which implies that $M(z, u, \varepsilon) \geq 1 - \varepsilon/2$. We conclude that $z = u$.

□

The following is an example where we can apply Theorem 2.1 but neither Theorem 1.1 nor Theorem 1.3.

Example 2.2. Let $(\mathbb{R}^+, M, \wedge)$ be the complete fuzzy metric space where $M(x, y, t) = 1$ if $|x - y| < t$ and $M(x, y, t) = 0$ if $|x - y| \geq t$, $x, y \in \mathbb{R}^+$, $t > 0$. (compare Example 1.2). Now, let T be the self mapping of $\mathbb{R}^+$ given by

$$Tx = \begin{cases} \frac{1}{2}, & \text{if } x \in [0, \frac{1}{2}], \\ 0, & \text{otherwise.} \end{cases}$$

We show that T is not a fuzzy Suzuki contraction on $(\mathbb{R}^+, M, \wedge)$.
Fix $\alpha \in (0, 1)$. For $t = 1/2$, $x = 1/2$ and $y = 3/4$ we get $|x - y| = 1/4 < t$ and $|x - Tx| = 0 < t$, so, $M(x, y, t) = M(x, Tx, t) = 1 > 1 - t$, but $|Tx - Ty| = 1/2 > \alpha t$, and thus, $M(Tx, Ty, \alpha t) = 0 < 1 - \alpha t$. Hence, we cannot apply Theorem 1.3.

Note also that for $x = 1$ and $y = 1/2$, we get $d(x, Tx) = 2d(x, y)$, and $d(Tx, Ty) = d(x, y)$, so that we cannot apply Theorem 1.1.

However, T is a w-fuzzy Suzuki contraction on $(\mathbb{R}^+, M, \wedge)$ as we show in the following.
Let

$$W : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to [0, 1]$$

defined as:

$W(x, y, 0) = 0$ for all $x, y \in \mathbb{R}^+$,

$W(x, y, t) = 1$ if $|x - y| < t$ and $x, y \in [0, 1/2]$, $t > 0$,

and

$W(x, y, t) = 0$, otherwise.

We check that W satisfies conditions (FW1)-(FW3), and thus it is a w-fuzzy distance on $(\mathbb{R}^+, M, \wedge)$.

(FW1): We only consider the case that $W(x, z, t) = W(z, y, s) = 1$. It follows that $|x - z| < t$, $|z - y| < s$, with $x, y, z \in [0, 1/2]$. Hence, $|x - y| < t + s$, which implies that $W(x, y, t + s) = 1$.

(FW2): Let $x \in \mathbb{R}^+$ and $y_n \to y$. We consider the case $\limsup_n W(x, y_n, t) = 1$. Pick $\varepsilon \in (0, t)$. Since $y_n \to y$ we deduce from the definition of W that $y \in [0, 1/2]$ and $|x - y| < t + \varepsilon$. Hence, $W(x, y, t + \varepsilon) = 1$.

(FW3): Given $\varepsilon \in (0,1)$ put $\delta = \varepsilon/2$. Suppose that $W(x,y,t) \geq 1-\delta$ and $W(x,z,s) \geq 1-\delta$. Then, $W(x,y,t) = W(x,z,s) = 1$, so $x,y,z \in [0,1/2]$ and $|y-z| < t+s$. Consequently, $M(y,z,t+s) = 1 > 1-\varepsilon$.

Finally, let $x,y \in \mathbb{R}^+$ and $t > 0$ such that $(x,y) \in SW(t)$. Then,

$$\min\{W(x,y,t), W(x,Tx,t)\} > 1-t.$$

We distinguish two cases:

- Case 1. $t \leq 1$. Since $W(x,y,t) > 1-t$, we deduce that $W(x,y,t) > 0$, i.e., $W(x,y,t) = 1$, so $|x-y| < t$ and $x,y \in [0,1/2]$. Consequently, $Tx = Ty = 1/2$ and, thus, $|Tx - Ty| = 0$. Hence, $W(Tx,Ty,t/2) = 1 > 1-t/2$.

- Case 2. $t > 1$. Since $Tx, Ty \in [0,1/2]$ we have $|Tx - Ty| \leq 1/2 < t/2$. It follows from the definition of W that $W(Tx,Ty,t/2) = 1$.

We have shown that T is a w-fuzzy Suzuki contraction with $\alpha = 1/2$, and, hence, all conditions of Theorem 2.1 are satisfied.

We finish with an example showing that Theorem 2.1 cannot be generalized to the case where the contraction condition

$$(x,y) \in WS(t) \Longrightarrow W(Tx,Ty,\alpha t) > 1-\alpha t,$$

$x,y \in X$ and $t > 0$, with

$$WS(t) := \{(x,y) \in X \times X : \min\{W(x,y,t), W(x,Tx,t)\} > 1-t\},$$

is replaced with the more general condition

$$(x,y) \in WS^*(t) \Longrightarrow W(Tx,Ty,\alpha t) > 1-\alpha t,$$

$x,y \in X$ and $t > 0$, where

$$WS^*(t) := \{(x,y) \in X \times X : W(x,y,t) * W(x,Tx,t) > 1-t\},$$

not even if the w-fuzzy distance is the fuzzy metric $(M,*)$.

Example 2.3. Let $X = \{0,1\}$ and let M be the fuzzy set in $X \times X \times \mathbb{R}^+$ defined as

$M(x,y,0) = 0$ for all $x,y \in X$,

$M(x,x,t) = 1$ for all $x \in X$ and $t > 0$,

$M(x,y,t) = 0$ for all $x,y \in X$ with $x \neq y$, and $0 < t \leq 1/2$,

$M(x,y,t) = 1/2$ for all $x,y \in X$ with $x \neq y$, and $1/2 < t \leq 1$,

and

$M(x,y,t) = 1$ for all $x,y \in X$ with $x \neq y$, and $1 < t$.

It is routine to check that $(X,M,*)$ is a complete fuzzy metric space for any continuous t-norm $*$.

Now, let T be the self-mapping of X given by $T0 = 1$ and $T1 = 0$. Hence, T has no fixed points. We show that, nevertheless, T satisfies the following contraction condition for $\alpha = 1/2$:

$$(x,y) \in MS^*(t) \Longrightarrow M(Tx,Ty,\alpha t) > 1-\alpha t,$$

$x, y \in X$ and $t > 0$, where

$$MS^*(t) := \{(x, y) \in X \times X : M(x, y, t) *_L M(x, Tx, t) > 1 - t\}.$$

Indeed, suppose that $(x, y) \in MS^*(t)$ with $x \neq y$.

If $0 < t \leq 1/2$ we have $M(x, y, t) *_L M(x, Tx, t) = 0 *_L 0 = 0$, which implies $t > 1$, a contradiction.

If $1/2 < t \leq 1$ we have $M(x, y, t) *_L M(x, Tx, t) = 1/2 *_L 1/2 = 0$, which implies $t > 1$, a contradiction.

If $t > 1$, we distinguish two cases:

- Case 1. $t > 2$. Then, $M(Tx, Ty, t/2) = 1 > 1 - t/2$.
- Case 2. $1 < t \leq 2$. Then, $M(Tx, Ty, t/2) = 1/2 > 1 - t/2$.

Finally, note that, as may be expected, T does not satisfy the contraction condition of Theorem 2.1 because for any $\alpha \in (0, 1)$, we have, taking $t = 1/2\alpha$,

$$\min\{M(x, y, t), M(x, Tx, t)\} \geq \frac{1}{2} > 1 - t,$$

but, for $x \neq y$, $M(Tx, Ty, \alpha t) = M(Tx, Ty, 1/2) = 0 < 1/2 = 1 - \alpha t$.

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Conflict of Interest

The author declares no conflict of interest.

References

[1] S. Al-Homidan, Q. H. Ansari, J. C. Yao, Some generalizations of Ekeland-type variational principle with applications to equilibrium problems and fixed point theory, *Nonlinear Anal.* **69** (2008), 126–139.

[2] C. Alegre, J. Marín, Modified w-distances on quasi-metric spaces and a fixed point theorem on complete quasi-metric spaces, *Topol. Appl.* **203** (2016), 32–41.

[3] J. Caristi, Fixed point theorems for mappings satisfying inwardness conditions, *Trans. Amer. Math. Soc.* **215** (1976), 241–251.

[4] I. Ekeland, On the variational principle, *J. Math. Anal. Appl.* **47** (1974), 324–353.

[5] O. Hadžić, E. Pap, *Fixed Point Theory in Probabilistic Metric Spaces*, Kluwer Academic Publishers, Dordrecht, 2001.

[6] T. L. Hicks, Fixed point theory in probabilistic metric spaces, *Rev. Res. Fac. Sci. Univ. Novi Sad* **13** (1983), 63–72.

[7] O. Kada, T. Suzuki, W. Takahashi, Nonconvex minimization theorems and fixed point theorems in complete metric spaces, *Math. Japon.* **44** (1996), 381–391.

[8] E. Karapınar, C. Chifu, Results in wt-distance over b-metric spaces, *Mathematics* **8** (2020), 220.

[9] E. Klement, R. Mesiar, E. Pap, *Triangular Norms*, Kluwer Academic Publishers, Dordrecht, 2000.

[10] D. Kocev, E. Karapınar, V. Rakočević, On quasi-contraction mappings of Ćirić and Fisher type via w-distance, *Quaest. Math.* **42** (2019), 1–14.

[11] I. Kramosil, J. Michálek, Fuzzy metrics and statistical metric spaces, *Kybernetika* **11** (1975), 336–344.

[12] H. Lakzian, V. Rakočević, H. Aydi, Extensions of Kannan contraction via w-distances, *Aequat. Math.* **93** (2019), 1231–1244.

[13] C. Mongkolkeha, Y. J. Cho, Some coincidence point theorems in ordered metric spaces via w-distance, *Carpathian J. Math.* **34** (2018), 207–214.

[14] S. Park, On generalizations of the Ekeland-type variational principles, *Nonlinear Anal.* **39** (2000), 881–889.

[15] V. Rakočević, *Fixed Point Results in W-Distance Spaces*, Monographs and Research Notes in Mathematics, CRC Press, Boca Raton, 2022.

[16] S. Romaguera, w-distances on fuzzy metric spaces and fixed points, *Mathematics* **8** (2020), 1909.

[17] S. Romaguera, Fuzzy contractions of Suzuki type and a characterization of fuzzy metric completeness, *J. Nonlinear Convex Anal.* **23** (2022), 1487–1494.

[18] S. Romaguera, P. Tirado, Presymmetric w-distances on metric spaces, *Mathematics* **12** (2024), 305.

[19] S. Romaguera, P. Tirado, The property of presymmetry for w-distances on quasi-metric spaces, *Filomat* **38** (2024), 8937–8945.

[20] R. Saadati, D. O'Regan, S. M. Vaezpour, J. K. Kim, Generalized distance and common fixed point theorems in Menger probabilistic metric spaces, *Bull. Iran. Math. Soc.* **35** (2009), 97–117.

[21] T. Shioji, T. Suzuki, W. Takahashi, Contractive mappings, Kannan mappings and metric completeness, *Proc. Amer. Math. Soc.* **126** (1998), 3117–3124.

[22] T. Suzuki, Several fixed point theorems in complete metric spaces, *Yokohama Math. J.* **44** (1997), 61–72.

[23] T. Suzuki, A generalized Banach contraction principle that characterizes metric completeness, *Proc. Amer. Math. Soc.* **136** (2008), 1861–1869.

[24] T. Suzuki, W. Takahashi, Fixed point theorems and characterizations of metric completeness, *Topol. Methods Nonlinear Anal.* **8** (1996), 371–382.

[25] W. Takahashi, Existence theorems generalizing fixed point theorems for multivalued mappings, in: *Fixed Point Theory and Applications*, Pitman Res. Notes Math. Ser., vol. 252, Longman Sci. Tech., 1991, pp. 397–406.

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